

CMC foliations of closed manifolds

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Abstract

We prove that every closed, smooth n -manifold X admits a Riemannian metric together with a CMC foliation if and only if its Euler characteristic is zero, where by a CMC foliation we mean a smooth, codimension-one, transversely oriented foliation with leaves of constant mean curvature and where the value of the constant mean curvature can vary from leaf to leaf. Furthermore, we prove that this CMC foliation of X can be chosen so that when $n \geq 2$, the constant values of the mean curvatures of its leaves change sign. We also prove a general structure theorem for any such non-minimal CMC foliation of X that describes relationships between the geometry and topology of the leaves, including the property that there exist compact leaves for every attained value of the mean curvature.

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1 Introduction.

This manuscript studies the existence, geometry and topology of smooth, transversely oriented foliations \mathcal{F} of a smooth closed Riemannian n -manifold X (not necessarily orientable), such that all of the leaves of \mathcal{F} are two-sided hypersurfaces of constant mean curvature and where the value of the constant mean curvature can vary from leaf to leaf; here, that the foliation is transversely oriented just means that there exists a smooth, unit vector field N on X that is normal to the leaves of the foliation, and the convention of signs for the mean curvature H of a leaf L of \mathcal{F} is

$$(n-1)H = \sum_{i=1}^{n-1} \langle \nabla_{E_i} E_i, N \rangle, \quad (1)$$

where \langle, \rangle is the ambient metric on X , ∇ is the Riemannian connection for the induced metric on L , and $\{E_1, \dots, E_{n-1}\}$ is a local orthonormal frame for the tangent bundle

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of L . Such a foliation \mathcal{F} is called a *CMC foliation* of X ; this is a particular case of a *tense* foliation, defined as a smooth foliation of X by k -dimensional submanifolds with parallel mean curvature vector, see e.g., Definition 1.36 in Rovenskii [17]. All manifolds and foliations appearing here will be assumed to be smooth (of class C^∞) unless otherwise stated.

By the next theorem, the vanishing of the Euler characteristic of a closed n -manifold is equivalent to the existence of a CMC foliation of the manifold with respect to some Riemannian metric. In the case the n -manifold is orientable, this theorem was proved by Oshikiri [13]; we emphasize that although in our presentation we give an alternative proof of the orientable case that uses a previous result by Oshikiri (Theorem 2.5 below), this approach is in fact not necessary, as we give another proof of Theorem 1.1 which does not use Oshikiri's results and which also covers the non-orientable case. Furthermore, when $n \geq 3$ the CMC foliations \mathcal{F} that we construct on a given closed n -manifold X with vanishing Euler characteristic satisfy that there are a finite number of components of the complement of the sublamination of minimal leaves in \mathcal{F} such that each of these foliated components is diffeomorphic to the product of an open $(n-1)$ -disk D and a circle \mathbb{S}^1 , with isometry group containing $SO(n-1) \times \mathbb{S}^1$; furthermore, the universal cover $D \times \mathbb{R}$ of each such component together with its lifted foliation and metric are equivalent to a rather explicit CMC foliation \mathcal{F}_n on $D \times \mathbb{R}$ with a product metric g_n , such that this structure is invariant under the action of $SO(n-1) \times \mathbb{R}$ and depends only on the dimension n .

Note that by definition, a CMC foliation is necessarily smooth.

Theorem 1.1 (Existence Theorem for CMC Foliations) *A closed n -manifold admits a CMC foliation for some Riemannian metric if and only if its Euler characteristic is zero. When $n \geq 2$, the CMC foliation can be taken to be non-minimal.*

Since closed (topological) three-manifolds admit smooth structures and the Euler characteristic of any closed manifold of odd dimension is zero, the previous theorem has the following corollary.

Corollary 1.2 *Every closed topological three-manifold admits a smooth structure together with a Riemannian metric and a non-minimal CMC foliation.*

Our proof of Theorem 1.1 is motivated by two seminal works. The first one, due to Thurston (Theorem 1(a) in [20]), shows that a necessary and sufficient condition for a smooth closed n -manifold X to admit a smooth, codimension-one foliation \mathcal{F} is for its Euler characteristic to vanish; for our applications, we will need to check that \mathcal{F} can be chosen to be transversely oriented. The second one is the result by Sullivan (Corollary 3 in [19]) that given such a pair (X, \mathcal{F}) where \mathcal{F} is orientable (see Definition 2.3), then X admits a smooth Riemannian metric g_X for which \mathcal{F} is a minimal foliation (i.e., \mathcal{F} is *geometrically taut*) if and only if for every compact leaf L of \mathcal{F} there exists a closed transversal that intersects L (i.e., \mathcal{F} is *homologically taut*); for our applications, we will need to prove that in the implication ‘homologically taut \Rightarrow geometrically taut’, Sullivan's hypothesis that the foliation \mathcal{F} be orientable can be removed.

Theorem 1.1 is also related to the question of prescribing a mean curvature function for a given transversely oriented, codimension-one foliation \mathcal{F} of a closed n -manifold X :

Walczak [21] asked the question of which smooth functions f on X can be written as mean curvature functions of the leaves of \mathcal{F} with respect to some Riemannian metric on X . In this line, Oshikiri [14, 15] characterized the solutions to this problem when X is orientable; in particular, under the hypothesis that X is orientable, he described for which functions $f \in C^\infty(X)$ that are constant along the leaves of \mathcal{F} , there exists a Riemannian metric on X that makes \mathcal{F} a CMC foliation (see Theorem 2.5 below for a precise statement of Oshikiri's result in the general setting).

An application of the divergence theorem shows that a geometrically taut foliation of a closed three-manifold X cannot have a Reeb component¹, since such a foliation can clearly not have a compact leaf that separates the manifold. Therefore, Novikov's theorem [11], which implies that any foliation of the three-sphere admits a Reeb component, also implies that the three-sphere does not admit any geometrically taut foliations. By work of Novikov [11] and Rosenberg [16] (also see Corollary 9.1.9 in Candel and Conlon [2]), every closed orientable three-manifold X admitting a codimension-one, transversely oriented foliation without Reeb components is either $\mathbb{S}^2 \times \mathbb{S}^1$ or irreducible²; in particular, such an X is a prime three-manifold. This fact together with the divergence theorem imply that a closed non-prime three-manifold X does not admit any geometrically taut foliations (i.e., such a manifold does not admit any foliations that are minimal for some Riemannian ambient metric); however, every closed three-manifold admits a Riemannian metric together with a CMC foliation by Corollary 1.2.

We now explain the organization of the paper. In Section 2 we cover some of the basic definitions related to CMC foliations. In Section 3 we will prove Theorem 1.1 in the case $n \leq 2$, so we will assume $n \geq 3$ for the remainder of the paper. In Section 4 we study the existence of codimension-one, $(SO(n-1) \times \mathbb{R})$ -invariant CMC foliations \mathcal{R}_{n-1} of the Riemannian product of the real number line \mathbb{R} with the closed unit $(n-1)$ -disk $\overline{\mathbb{D}}(1) \subset \mathbb{R}^{n-1}$ with respect to a certain $SO(n-1)$ -invariant metric, whose leaves are of one of two types: the leaves that intersect $\mathbb{D}(r_1) \times \mathbb{R}$ (here $\mathbb{D}(r_1) = \{x \in \mathbb{R}^{n-1} \mid \|x\| < r_1\}$ and $0 < r_1 < 1$) are rotationally symmetric hypersurfaces which are graphical over $\mathbb{D}(r_1) \times \{0\}$ and asymptotic to the vertical $(n-1)$ -cylinder $\mathbb{S}^{n-2}(r_1) \times \mathbb{R}$; the remaining leaves of \mathcal{R}_{n-1} are the vertical cylinders $\mathbb{S}^{n-2}(r) \times \mathbb{R}$, $r \in [r_1, 1]$. All leaves of \mathcal{R}_{n-1} in $\mathbb{D}^{n-1}(r_1) \times \mathbb{R}$ are vertical translates of a single such leaf (in particular, they all have the same constant mean curvature, equal to the constant value of the mean curvature of $\mathbb{S}^{n-2}(r_1) \times \mathbb{R}$), while the (constant) mean curvature values of the cylinders $\mathbb{S}^{n-2}(r) \times \mathbb{R}$, $r \in [r_1, 1]$, vary from leaf to leaf. These foliations \mathcal{R}_{n-1} give rise under the quotient action of $\mathbb{Z} \subset \mathbb{R}$ to what we call *enlarged foliated Reeb components* $\mathcal{R}_{n-1}/\mathbb{Z}$, that are diffeomorphic to $\overline{\mathbb{D}}(1) \times \mathbb{S}^1$.

Section 5 will be devoted to proving Theorem 1.1 along the following lines. The sufficient implication follows directly from the divergence theorem and the Poincaré-Hopf index theorem. As for the necessary implication, the results in [20] imply that a smooth, closed n -manifold X with Euler characteristic zero admits a smooth, transversely oriented foliation \mathcal{F}' of codimension one. After a simple modification of \mathcal{F}' by the classical technique of tubularization (explained in Subsection 5.1), \mathcal{F}' can be

¹Classical Reeb components of foliations of closed three-manifolds are defined at the beginning of Section 4. In Definition 4.5 we will describe the notion of *enlarged Reeb component* of a codimension-one foliation, which makes sense in all dimensions and differs for the classical definition in the case $n = 3$ as it does not contain a single compact leaf.

²A three-manifold X is *irreducible* if every embedded two-sphere in X bounds a three-ball in X .

assumed to have at least one non-compact leaf. In Section 5.2 we will prove the existence of a finite collection $\Delta = \{\gamma_1, \dots, \gamma_k\}$ of pairwise disjoint, compact embedded arcs in X that are transverse to the leaves of \mathcal{F}' and such that every compact leaf of the foliation intersects at least one of these arcs; this existence result will follow from work of Haefliger [5] on the compactness of the set of compact leaves of any codimension-one foliation of X . We will then proceed to modify \mathcal{F}' using again turbularization by introducing pairs of what we called in the previous paragraph “enlarged Reeb components”, one pair of these enlarged Reeb components for each $\gamma_i \in \Delta$. These modifications give rise to a new transversely oriented foliation \mathcal{F} and a related function $f \in C^\infty(X)$ that is constant along the leaves of \mathcal{F} and that when X is orientable, satisfies Oshikiri’s condition [14, 15] (see also Theorem 2.5 below) for there to exist a Riemannian metric on X that makes \mathcal{F} a CMC foliation. Since the function f also changes sign on X , Theorem 2.5 will produce the ambient metric on X such that the foliation \mathcal{F} satisfies the properties stated in Theorem 1.1 when X is orientable. This analysis of the orientable case for X will be done in Section 5.3. In Section 5.4 we will give a direct proof of the necessary implication of Theorem 1.1 that avoids the results of Oshikiri and also works when the manifold X is non-orientable; this direct proof depends on the rotationally invariant foliations constructed in Section 4, Theorem 2 in Moser [10], as well as on a generalization of Sullivan’s theorem to the case of non-orientable codimension-one foliations, given in Theorem 5.3 of Section 5.5.

Finally, in Section 6 we will prove the Structure Theorem 1.3 given below on the geometry and topology of non-minimal CMC foliations of a closed n -manifold. Before stating this theorem, we fix some notation. For a CMC foliation \mathcal{F} of a (connected) closed Riemannian n -manifold X :

- $N_{\mathcal{F}}$ denotes the unit normal vector field to \mathcal{F} whose direction coincides with the given transverse orientation.
- $H_{\mathcal{F}}: X \rightarrow \mathbb{R}$ stands for the *mean curvature function* of \mathcal{F} with respect to $N_{\mathcal{F}}$.
- $H_{\mathcal{F}}(X) = [\min H_{\mathcal{F}}, \max H_{\mathcal{F}}]$ is the image of $H_{\mathcal{F}}$.
- $\mathcal{C}_{\mathcal{F}}$ denotes the union of the compact leaves in \mathcal{F} , which is a compact subset of X by the aforementioned result of Haefliger [5].

Theorem 1.3 (Structure Theorem for CMC Foliations) *Let (X, g) be a closed connected Riemannian n -manifold which admits a non-minimal CMC foliation \mathcal{F} . Then:*

1. $\int_X H_{\mathcal{F}} dV = 0$ and so, $H_{\mathcal{F}}$ changes sign (here dV denotes the volume element with respect to g).
2. For H a regular value of $H_{\mathcal{F}}$, $H_{\mathcal{F}}^{-1}(H)$ consists of a finite number of compact leaves of \mathcal{F} contained in $\text{Int}(\mathcal{C}_{\mathcal{F}})$.
3. $X - \mathcal{C}_{\mathcal{F}}$ consists of a countable number of open components and the leaves in each such component Δ have the same mean curvature as the finite positive number of compact leaves in $\partial\Delta$; furthermore, every leaf in the closure of $X - \mathcal{C}_{\mathcal{F}}$ is stable. In particular, except for a countable subset of $H_{\mathcal{F}}(X)$, every leaf of \mathcal{F} with mean curvature H is compact, and for every $H \in H_{\mathcal{F}}(X)$, there exists at least one compact leaf of \mathcal{F} with mean curvature H .

4. a. Suppose that L is a leaf of \mathcal{F} that contains a regular point of $H_{\mathcal{F}}$. Then L is compact, it consists entirely of regular points of $H_{\mathcal{F}}$ and lies in the interior of $\mathcal{C}_{\mathcal{F}}$. Furthermore, L has index³ zero if and only if the function $g(\nabla H_{\mathcal{F}}, N_{\mathcal{F}}) = N_{\mathcal{F}}(H_{\mathcal{F}})$ is negative along L , and if the index of L is zero, then it also has nullity zero.
- b. Suppose that L is a leaf of \mathcal{F} that is disjoint from the regular points of $H_{\mathcal{F}}$. Then the index of L is zero, and if L is a limit leaf⁴ of the CMC lamination of X consisting of the compact leaves of \mathcal{F} , then L is compact with nullity one.
5. Any leaf of \mathcal{F} with mean curvature equal to $\min H_{\mathcal{F}}$ or $\max H_{\mathcal{F}}$ is stable and such a leaf can be chosen to be compact with nullity one.

Remark 1.4 Let (X, g) and \mathcal{F} be as in Theorem 1.3.

- (i) Item 5 in Theorem 1.3 implies restrictions on the Ricci curvature of (X, g) ; namely, $\text{Ric}(N_{\mathcal{F}}) \leq -(n-1) \max H_{\mathcal{F}}^2$ at some point in X , which can be obtained from evaluating the index form of the Jacobi operator of a compact (stable) leaf with mean curvature equal to either $-\min H_{\mathcal{F}}$, or $\max H_{\mathcal{F}}$, at the constant function one on the leaf.
- (ii) In the case that $n = 3$, X is orientable and \mathcal{F} is not topologically a product foliation of $X = \mathbb{S}^2 \times \mathbb{S}^1$ by spheres, then item 1 in Theorem 2.13 of [8] and item 5 in Theorem 1.3 imply that the scalar curvature of (X, g) cannot be everywhere greater than $-\frac{2}{3} \max H_{\mathcal{F}}^2$.

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2 Preliminaries.

Definition 2.1 A smooth codimension-one *lamination* of a Riemannian n -manifold X is the union of a collection of pairwise disjoint, connected, injectively immersed hypersurfaces, with a certain local product structure. More precisely, it is a pair $(\mathcal{L}, \mathcal{A})$ satisfying:

1. \mathcal{L} is a closed subset of X ;
2. $\mathcal{A} = \{\varphi_{\beta}: \mathbb{D} \times (0, 1) \rightarrow U_{\beta}\}_{\beta}$ is a maximal⁵ atlas of (smooth) coordinate charts of X (here \mathbb{D} is the open unit disk in \mathbb{R}^{n-1} , $(0, 1)$ is the open unit interval in \mathbb{R} and U_{β} is an open subset of X). Charts in \mathcal{A} are called *lamination charts*.
3. For each β , there exists a closed subset C_{β} of $(0, 1)$ such that $\varphi_{\beta}^{-1}(U_{\beta} \cap \mathcal{L}) = \mathbb{D} \times C_{\beta}$.

³This index (resp. nullity) is the number of negative eigenvalues (resp. multiplicity of zero as an eigenvalue) of the Jacobi operator of L viewed as a compact, two-sided hypersurface with constant mean curvature in X .

⁴See Definition 2.2 for the definition of a limit leaf of a CMC lamination.

⁵Here maximality refers to atlases satisfying properties 2 and 3.

We will simply denote laminations by \mathcal{L} , omitting the lamination charts φ_β unless explicitly necessary. A smooth lamination \mathcal{L} is said to be a *foliation of X* if $\mathcal{L} = X$ (and the corresponding charts in \mathcal{A} are called *foliation charts*). Every lamination \mathcal{L} decomposes into a collection of disjoint, connected smooth hypersurfaces (locally given by $\varphi_\beta(\mathbb{D} \times \{t\})$, $t \in C_\beta$, with the notation above), called the *leaves* of \mathcal{L} . Note that if $\Delta \subset \mathcal{L}$ is any collection of leaves of \mathcal{L} , then the closure of the union of these leaves has the structure of a lamination within \mathcal{L} , which we will call a *sublamination*.

A smooth codimension-one lamination \mathcal{L} of X is said to be a *CMC lamination* if each of its leaves has constant mean curvature (possibly varying from leaf to leaf). Given $H \in \mathbb{R}$, an *H -lamination* of X is a CMC lamination all whose leaves have the same mean curvature H . If $H = 0$, the H -lamination is called a *minimal lamination*. Observe that a closed hypersurface is a particular case of lamination, hence H -laminations are natural generalizations of (closed) H -hypersurfaces.

Definition 2.2 Given a smooth codimension-one lamination \mathcal{L} of a Riemannian n -manifold X , a point $p \in \mathcal{L}$ is a *limit point* if there exists a coordinate chart $\varphi_\beta: \mathbb{D} \times (0, 1) \rightarrow U_\beta$ as in Definition 2.1 such that $p \in U_\beta$ and $\varphi_\beta^{-1}(p) = (x, t)$ with t belonging to the accumulation set of C_β . It is easy to show that if p is a limit point of a codimension-one lamination \mathcal{L} (resp. of a H -lamination), then the leaf L of \mathcal{L} passing through p consists entirely of limit points of \mathcal{L} ; in this case, L is called a *limit leaf* of \mathcal{L} .

Definition 2.3 Let \mathcal{F} be a codimension-one foliation of a manifold X . \mathcal{F} is called *transversely orientable* if there exists a continuous, nowhere zero vector field whose integral curves intersect transversely to the leaves of \mathcal{F} . Once such a vector field has been chosen, we call \mathcal{F} a *transversely oriented* codimension-one foliation. \mathcal{F} is called *orientable* if there exists a smooth $(n - 1)$ -form on X whose restriction to the tangent spaces of leaves of \mathcal{F} is never zero. These notions can be expressed in terms of the tangent bundle $T\mathcal{F}$ and the normal bundle $T^\perp\mathcal{F}$ to \mathcal{F} (once we have chosen a Riemannian metric in X): \mathcal{F} is orientable (resp. transversely orientable) when $T\mathcal{F}$ (resp. $T^\perp\mathcal{F}$) is orientable.

Definition 2.4 Let \mathcal{F} be a transversely oriented, codimension-one foliation of an n -manifold X and let γ be a smooth simple closed curve contained in a leaf L of \mathcal{F} . After picking a Riemannian metric g on X , we can define a *normal fence* above γ as follows. Consider exponential coordinates for (X, g) along γ . The *right* normal fence is the set

$$A = \{\exp_{\gamma(t)}(sN_{\mathcal{F}}(\gamma(t))) \mid t \in \mathbb{S}^1, s \in [0, \varepsilon]\},$$

where $N_{\mathcal{F}}$ is the positive unit normal vector to \mathcal{F} and $\varepsilon > 0$ is sufficiently small so that A defines an embedded annulus in X . If the parameter s moves in $[-\varepsilon, 0]$, then we call the annulus a *left* normal fence.

We finish this section by stating Oshikiri's theorem [15] about which smooth functions on a compact oriented n -manifold X can be viewed as mean curvature functions of a given transversely oriented, codimension-one foliation \mathcal{F} on X . To state this result, we first need some notation. A domain $D \subset X$ is called *saturated* if it is a union of leaves of \mathcal{F} . A compact, smooth saturated domain $D \subset X$ is called a *(+)-foliated compact domain* ((+)-fcd, for short) if the transverse orientation of \mathcal{F} points outward

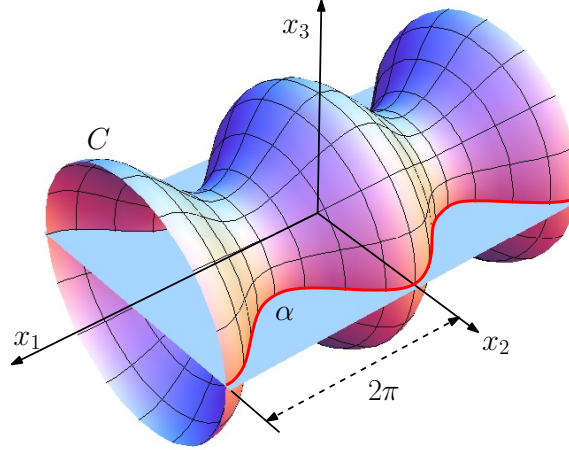


Figure 1: The circles in C foliate the surface by curves of constant geodesic curvature. The symmetry R is the composition of the reflection in the (x_1, x_2) -plane (depicted in the figure) with the translation by the vector $(2\pi, 0, 0)$.

everywhere on ∂D , and we call D a $(-)$ -foliated compact domain (or $(-)$ -fcd) if the transverse orientation of \mathcal{F} is inward pointing everywhere on ∂D . A smooth function $f: X \rightarrow \mathbb{R}$ is called *admissible for \mathcal{F}* if there exists a Riemannian metric on X such that f is the oppositely signed mean curvature function of the leaves of \mathcal{F} with respect to the given transversal orientation. In other words for each $x \in X$, $-f(x)$ is the mean curvature H (defined as in equation (1)) of the leaf L_x of \mathcal{F} passing through x with respect to the unit normal vector field to L_x whose direction coincides with the given transverse orientation.

Theorem 2.5 ([15]) *Let \mathcal{F} be a codimension-one, transversely oriented foliation in a compact oriented n -manifold X , such that \mathcal{F} contains at least one $(+)$ -fcd. Then, a smooth function $f: X \rightarrow \mathbb{R}$ is admissible for \mathcal{F} if and only if every minimal⁶ $(+)$ -fcd contains a point where f is positive, and every $(-)$ -fcd contains a point where f is negative.*

Note that if a foliation \mathcal{F} as in Theorem 2.5 does not contain any $(+)$ -fcd, then it also does not contain any $(-)$ -fcd and thus Corollary 6.3.4 in Candel and Conlon [1] implies that \mathcal{F} is homologically taut.

3 Proof of Theorem 1.1 in the case $n \leq 2$.

Consider the curve $\alpha = \{(t, 3 + \cos t) \mid t \in \mathbb{R}\}$ in the (x_1, x_2) -plane and let C in \mathbb{R}^3 be the surface obtained by revolving α around the x_1 -axis. Let \mathcal{F} be the foliation of C by circles contained in planes orthogonal to the x_1 -axis, whose leaves have constant geodesic curvature, see Fig. 1. \mathcal{F} is transversely oriented by the normal vectors to the circles in C that have positive inner product in \mathbb{R}^3 with ∂_{x_1} . Since the map $R(x_1, x_2, x_3) =$

⁶Here, *minimal* refers to the partial order given by inclusion.

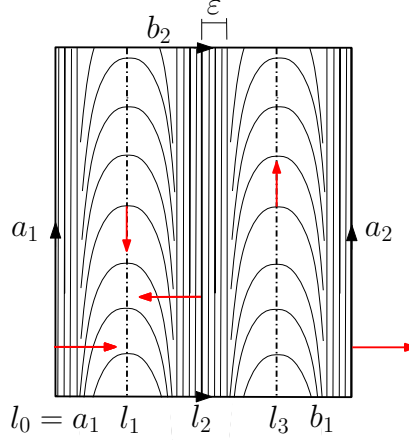


Figure 2: The arrows indicate the transverse orientation on the foliation \mathcal{F} .

$(2\pi + x_1, x_2, -x_3)$ preserves the orientation of the CMC foliation, then \mathcal{F} descends to a CMC foliation of the Klein bottle C/R or to the torus $C/(R^2)$. By classification of closed surfaces, a closed surface with Euler characteristic zero must be a torus or a Klein bottle. Thus, Theorem 1.1 trivially holds when $n = 2$.

In the remainder of this section we will produce a Riemannian metric g on a two-dimensional torus M together with a one-dimensional, transversely oriented foliation \mathcal{F} by curves of constant geodesic curvature, where the geodesic curvature function $H_{\mathcal{F}}$ of the foliation changes sign and \mathcal{F} contains two Reeb components. We will construct such a metric g with a one-dimensional isometry group whose elements leave invariant \mathcal{F} . The construction will be made so that both \mathcal{F} and g induce similar objects on a quotient Klein bottle.

Consider a rectangle R with sides a_1, a_2, b_1, b_2 identified as usual to produce a flat torus M after identification of its sides; assume that the length of a_1, a_2 is 1. Consider vertical lines l_0, \dots, l_3 as in Fig. 2. Let G be the group of isometries of the flat metric on M generated by the reflections across l_0, l_1 and by the \mathbb{S}^1 -family of vertical translations; as a group, G is isomorphic to the product of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{S}^1$.

We now describe the foliation \mathcal{F} of M depicted in Fig. 2. Take $\varepsilon > 0$ sufficiently small so that the closed vertical strips of width ε around the lines l_0 and l_2 are pairwise disjoint. \mathcal{F} restricts to the union S of these strips as the product foliation by vertical lines; the restriction of \mathcal{F} to each of the open strips in the complement of S consists of one-dimensional Reeb foliations invariant under G , see Fig. 2. The transverse orientation we choose for \mathcal{F} is described in the figure.

Consider a smooth function $f: M \rightarrow [-1, 1]$ with the following properties.

- (F1) f is constant -1 (resp. $+1$) on the component of $M - S$ that contains l_1 (resp. l_3).
- (F2) f is invariant under the \mathbb{S}^1 -action by vertical translations, and under the reflections in l_1, l_3 .

(F3) If ψ is the reflection in either l_0 or l_2 , then $f \circ \psi = -f$.

By Theorem 2.5, f is an admissible function for \mathcal{F} in the sense explained at the end of Section 2. However, we are interested in obtaining the following more specific result. Given $i = 0, \dots, 3$, we let R_i denote the reflection in M across l_i . Let T_θ be the vertical translation by $\theta \in [0, 1)$ in M (recall that the length of $a_1 = 1$).

Theorem 3.1 *There exists a metric g on M that is invariant under the group G such that the function $-f$ is the geodesic curvature function of the foliation \mathcal{F} . In particular, for $I = T_{\frac{1}{2}} \circ R_1$, the quotient Klein bottle M/I with its quotient metric admits the transversely oriented CMC foliation \mathcal{F}/I .*

Proof. The key result used to prove Theorem 2.5 stated above is Oshikiri's Main Theorem in [12], and our proof will also be based on this Main Theorem. First note that $\int_M f dV = 0$, where dV is the volume form on M associated to the flat metric (with the usual orientation induced from the one of \mathbb{R}^2), and so there exists a 2-form ω' such that $d\omega' = f dV$. Since $\int_M f dV = 0$ and the other homological condition in item 3 of Oshikiri's Main Theorem in [12] is satisfied for \mathcal{F} , then the application of the Hahn-Banach theorem in the proof of the Oshikiri's Main Theorem guarantees the existence of a 1-form ω_1 on M such that $d\omega_1 = f dV$ and ω_1 restricted to any leaf of \mathcal{F} is a nowhere zero one-form on the leaf. Note that in order to verify that item 3 of Oshikiri's Main Theorem holds, it is easier to check that the following equivalent condition (3*) given on page 515 in [14] holds:

- (i) $\int_M f dV = 0$, and
- (ii) $\int_D f dV > 0$ for any (+)-fed D .

We next explain how to modify ω_1 to obtain another 1-form ω on M that satisfies:

1. $T_\theta^*(\omega) = \omega$, for each $\theta \in [0, 1)$.
2. $R_i^*(\omega) = (-1)^{i+1}\omega$, $i = 0, \dots, 3$.
3. $d\omega = f dV$ and ω restricted to any leaf of \mathcal{F} is a nowhere zero one-form on the leaf.

To obtain the desired 1-form, first average ω_1 with respect to the \mathbb{S}^1 -action given by the vertical translations to obtain a 1-form ω_2 ; in other words, $\omega_2(v_p) = \int_0^1 (T_\theta^* \omega_1)(v_p) d\theta$ for each tangent vector $v_p \in T_p M$; note that ω_2 also satisfies condition 3. Next successively define

$$\omega_3 = \frac{1}{2}(\omega_2 + R_1^* \omega_2), \quad \omega = \frac{1}{2}(\omega_3 - R_2^* \omega_3).$$

We write the flat metric g_0 on M as $g_0 = g^1 \oplus g^2$, where we are using the g_0 -orthogonal decomposition of TM by the normal and tangent subbundles $T^\perp \mathcal{F}$ and $T\mathcal{F}$ with respect to g_0 , where $g^1 = g_0|_{T^\perp \mathcal{F}}$ and $g^2 = g_0|_{T\mathcal{F}}$. Then, we can express $\omega_p|_{T_p \mathcal{F}}$ as a multiple of the length one-form dl_{g^2} associated to g^2 , namely $\omega_p = \omega_p(JN_p) dl_{g_0}$, where N_p is the unit normal vector (with respect to g_0) given by the transverse orientation to

\mathcal{F} , and J is the almost complex structure on M given by its orientation. Consider the metric on M given by

$$g = \frac{1}{\omega(JN)^2} g^1 \oplus \omega(JN)^2 g^2$$

with respect to the same decomposition of TM . It follows that the volume form associated to g is dV , and the restriction of ω to $T\mathcal{F}$ coincides with the length one-form of the restriction of g to $T\mathcal{F}$. Under this definition of metric on M and applying the so called Rummmler's calculation (see the beginning of Section 5.5 for more details), we conclude that $-f$ is the geodesic curvature function of the metric g . \square

Remark 3.2 Theorem 3.1 has an n -dimensional generalization that allows one to construct metrics on $\mathbb{S}^{n-1} \times \mathbb{S}^1$, where the two vertical rectangles in Fig. 2 bounded by l_0, l_2 are each replaced by the product of an $(n-1)$ -dimensional 'horizontal' disk D , $n \geq 3$, with a 'vertical' circle \mathbb{S}^1 ; these two domains $D \times \mathbb{S}^1$ are foliated by enlarged Reeb components (see Definition 4.2 in the next section), which are invariant under the action of vertical translations and reflections about vertical hyperplanes passing through l_1 and l_3 . In the next section we will construct such CMC foliations so that the ambient metric is a product metric on $D \times \mathbb{S}^1$, for some metric on D ; note that this type of metric in the case when $n = 2$ would necessarily be flat, which prevents the construction in Section 4 to work in dimension $n = 2$.

4 Rotationally symmetric H -foliations of Reeb type.

From this point on, we will always assume that $n \geq 3$.

Definition 4.1 Let $\mathbb{D}(R)$ the open disk of radius $R > 0$ in \mathbb{R}^{n-1} . For $z \in \mathbb{R}$, let $\Sigma_z \subset \mathbb{R}^{n-1} \times \mathbb{R}$ be the graph of the function

$$f + z: \mathbb{D}(1) \rightarrow \mathbb{R}, \quad f(x) = -\sec\left(\frac{\pi}{2}\|x\|^2\right), \quad x \in \mathbb{D}(1),$$

and let Σ_∞ be the cylinder $\partial\mathbb{D}(1) \times \mathbb{R} \subset \mathbb{R}^n$.

1. Consider the foliated closed cylinder $\overline{\mathbb{D}}(1) \times \mathbb{R}$ with leaves $\Sigma_z, z \in (-\infty, \infty]$. Up to diffeomorphism, any such foliation is called a *Reeb-type foliation* of the cylinder.
2. After passing to the quotient by the natural action of \mathbb{Z} acting on $\overline{\mathbb{D}}(1) \times \mathbb{R}$ by integer translations in the n -th variable, we obtain a compact n -manifold with boundary $(\overline{\mathbb{D}}(1) \times \mathbb{R})/\mathbb{Z}$; we call this foliated manifold a *Reeb-type foliation* of the solid torus $\overline{\mathbb{D}} \times \mathbb{S}^1$.
3. Given a foliation \mathcal{F} of an n -manifold X , a compact saturated domain $R \subset X$ is called a *Reeb component* of \mathcal{F} if it is diffeomorphically equivalent to a Reeb-type foliation of $\overline{\mathbb{D}} \times \mathbb{S}^1$.

In this section we will construct a smooth Reeb type foliation enlarged by a product foliation (see (E1), (E2) below) of a solid n -dimensional torus together with a smooth, rotationally symmetric ambient metric so that the leaves of the foliation have constant mean curvature, and torus leaves sufficiently close to the boundary of the solid cylinder

are minimal; by torus leaves, we mean leaves that are diffeomorphic to $\mathbb{S}^{n-2} \times \mathbb{S}^1$. The existence of some CMC foliation on the Reeb torus follows easily from Oshikiri's Theorem 2.5, although we will construct it explicitly by imposing rotational symmetry, as the analysis of the corresponding ODE could be interesting on its own right.

We consider on $\overline{\mathbb{D}} = \overline{\mathbb{D}}(1)$ polar coordinates $r \in [0, 1]$, $\theta \in \mathbb{S}^{n-2}(1)$. Given $H > 0$, our goal is to construct an *enlarged Reeb-type foliation* of the cylinder $\overline{\mathbb{D}} \times \mathbb{R}$; more precisely, we will construct a smooth, $SO(n-1)$ -invariant metric ds^2 over $\overline{\mathbb{D}}$ and a (smooth) $SO(n-1)$ -invariant, CMC foliation \mathcal{F} of the Riemannian product $(\overline{\mathbb{D}} \times \mathbb{R}, ds^2 + dz^2)$, satisfying the following conditions:

- (E1) There exists $r_1 \in (0, 1)$ such that the leaves of \mathcal{F} in $\mathbb{D}(r_1) \times \mathbb{R}$ are of the form Σ_z , $z \in \mathbb{R}$, where Σ_z is the vertical translate by $z \in \mathbb{R}$ of a smooth, $SO(n-1)$ -invariant H -hypersurface Σ which is a graph over $\mathbb{D}(r_1) \times \{0\}$. Furthermore, the generating profile curve $\Gamma \subset \{(r, z) \mid r \in [0, 1], z \in \mathbb{R}\}$ of Σ under the action of $SO(n-1)$ can be globally parameterized as $\Gamma(r) = (r, z(r))$, with $z: [0, r_1) \rightarrow \mathbb{R}$ being a smooth function that satisfies $z'(0) = 0$, $z'(r) > 0$ for all $r \in (0, r_1)$ and $z(r) \rightarrow +\infty$ as $r \rightarrow r_1^-$. Note that the restriction of \mathcal{F} to $\mathbb{D}(r_1) \times \mathbb{R}$ is an H -foliation with respect to the ambient metric $ds^2 + dz^2$.
- (E2) The restriction of \mathcal{F} to $(\overline{\mathbb{D}} - \mathbb{D}(r_1)) \times \mathbb{R}$ is the product foliation by vertical $(n-1)$ -dimensional cylinders $(\partial\mathbb{D}(R)) \times \mathbb{R}$, $R \in [r_1, 1]$. These cylinders also have constant mean curvature, which will vary from the value H at $R = r_1$ to the value zero for all $R \in [r_2, 1]$ (here $r_2 \in (r_1, 1)$). Note that the restriction of \mathcal{F} to $\overline{\mathbb{D}}(r_1) \times \mathbb{R}$ is a Reeb-type foliation in the sense of item 1 of Definition 4.1.

Definition 4.2 After taking quotients by the natural action of \mathbb{Z} by translations in the vertical factor of $\overline{\mathbb{D}} \times \mathbb{R}$, the quotient foliation of $\overline{\mathbb{D}} \times \mathbb{S}^1$ will be called an *enlarged Reeb-type foliation* of $\overline{\mathbb{D}} \times \mathbb{S}^1$, see Fig. 3.

The following analysis of the ODE system that corresponds to a rotationally symmetric H -hypersurface in $\overline{\mathbb{D}} \times \mathbb{R}$ is inspired by the paper [4] by Figueroa, Mercuri and Pedrosa. Every smooth, rotationally symmetric Riemannian metric on $\overline{\mathbb{D}}$ can be written in the form

$$ds^2 = dr^2 + \phi(r)^2 d\theta^2, \quad (2)$$

where $d\theta^2$ denotes the metric of sectional curvature 1 on $\mathbb{S}^{n-2}(1)$ and $\phi: [0, 1] \rightarrow [0, \infty)$ is a smooth function that satisfies $\phi(r) > 0$ if $r \in (0, 1]$, $\phi'(0) = 1$ and all whose derivatives of even order (including order zero) vanish at $r = 0$. Choose local coordinates $\theta_1, \dots, \theta_{n-2}$ on $\mathbb{S}^{n-2}(1)$ so that $\{\partial_{\theta_i} = \frac{\partial}{\partial \theta_i} \mid i = 1, \dots, n-2\}$ is an orthonormal basis of $(T\mathbb{S}^{n-2}(1), d\theta^2)$. Note that with respect to ds^2 , $\partial_r = \frac{\partial}{\partial r}$ is unitary and orthogonal to each ∂_{θ_i} , and that $|\partial_{\theta_i}| = \phi$. The $(n-2)$ -dimensional sphere $\mathbb{S}^{n-2}(r)$ of radius $r \in (0, 1]$ in $(\overline{\mathbb{D}}, ds^2)$ is totally umbilic with constant mean curvature

$$\kappa_r = \frac{\phi'(r)}{\phi(r)} \quad (3)$$

with respect to the inner pointing unit normal vector $-\partial_r$.

We consider on $\overline{\mathbb{D}} \times \mathbb{R}$ the product metric $g = ds^2 + dz^2$, where $z \in \mathbb{R}$ represents height in the \mathbb{R} -factor. The group $SO(n-1)$ acts on $(\overline{\mathbb{D}} \times \mathbb{R}, g)$ by isometries (rotations

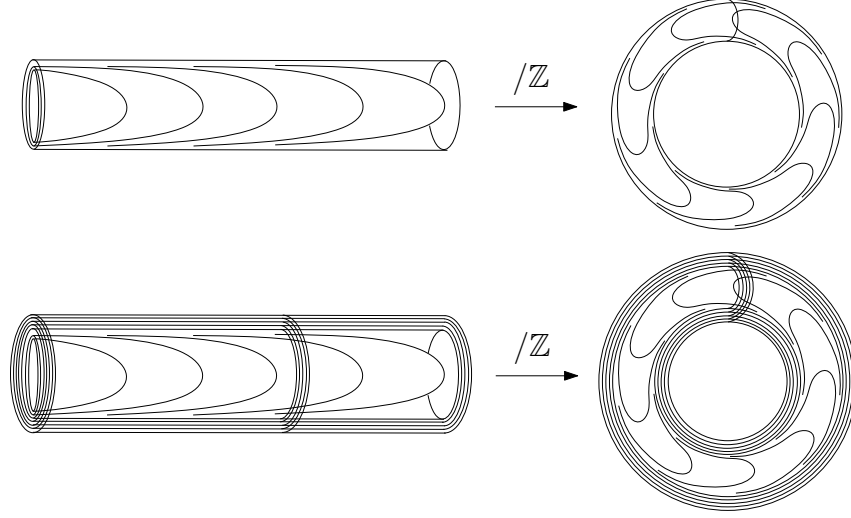


Figure 3: Above: A Reeb-type foliation of a cylinder (left) and of $\mathbb{D} \times \mathbb{S}^1$ (right). Below: An enlarged Reeb-type foliation of a cylinder (left) and of $\mathbb{D} \times \mathbb{S}^1$ (right).

about the z -axis $\{r = 0\}$), with orbit space $\mathcal{B} = \{(r, z) \in \mathbb{R}^2 \mid 0 \leq r \leq 1\}$, and the induced metric by g on $\mathcal{B}^* = \{(r, z) \mid 0 < r \leq 1\}$ is flat. Suppose that $s \mapsto \Gamma(s) = (r(s), z(s)) \in \mathcal{B}^*$ is a smooth curve parameterized by arc length. Let us denote by $\sigma(s)$ the angle that the velocity vector $\dot{\Gamma}(s)$ makes with the ∂_r direction. The planar curvature of Γ is $\kappa_\Gamma(s) = \dot{\sigma}(s)$, and the Frenet dihedron of Γ can be written in coordinates with respect to the orthonormal basis $\{\partial_r, \partial_z\}$ as

$$t = t(s) = (\dot{r}, \dot{z}) = (\cos \sigma, \sin \sigma), \quad n = n(s) = (-\sin \sigma, \cos \sigma).$$

Let Σ be the smooth hypersurface of $\mathbb{D}^* \times \mathbb{R}$ generated by Γ after the action of $SO(n-1)$ about the z -axis, where $\mathbb{D}^* = \mathbb{D} - \{0\}$. We want to relate the mean curvature function \mathcal{H} of Σ with κ_Γ and κ_r . As Σ is $SO(n-1)$ -invariant, we can do this by computing the mean curvature of Σ at a point $p \in \Sigma \cap \{(r, \vec{0}, z) \mid 0 < r \leq 1\}$, where $\vec{0}$ is the origin in \mathbb{R}^{n-3} . After a slight abuse of notation, we will identify $\Sigma \cap \{(r, \vec{0}, z) \mid 0 < r \leq 1\}$ with Γ . The tangent space $T_p \Sigma$ at such a point $p \in \Sigma$ admits the orthonormal basis $\{X_i = \frac{1}{\phi} \partial_{\theta_i} \mid i = 1, \dots, n-2\} \cup \{\dot{\Gamma}\}$ and thus,

$$(n-1)\mathcal{H} = \sum_{i=1}^{n-2} g(\nabla_{X_i} X_i, N) + g(\nabla_{\dot{\Gamma}} \dot{\Gamma}, N), \quad (4)$$

where ∇ is the Riemannian connection of g and N stands for the unit normal vector to Σ that coincides with n at p (the mean curvature function \mathcal{H} is computed with respect to N). As X_i is a horizontal vector field on the Riemannian product $(\mathbb{D}^* \times \mathbb{R}, g)$, then $\nabla_{X_i} X_i$ is also horizontal and given by $\nabla_{X_i} X_i = (\nabla_{X_i}^{\mathbb{D}} X_i, 0)$, where $\nabla^{\mathbb{D}}$ is the Riemannian connection in (\mathbb{D}, ds^2) . Since $N = n = -\sin \sigma \partial_r + \cos \sigma \partial_z$, then

$$\sum_{i=1}^{n-2} g(\nabla_{X_i} X_i, N) = -\sin \sigma \sum_{i=1}^{n-2} ds^2(\nabla_{X_i}^{\mathbb{D}} X_i, \partial_r) = (n-2)\kappa_r \sin \sigma \stackrel{(3)}{=} (n-2) \frac{\phi'(r)}{\phi(r)} \sin \sigma,$$

where $r = r(p)$. As for the last term of (4), since the vertical plane $\{y = 0\}$ is totally geodesic in $(\overline{\mathbb{D}} \times \mathbb{R}, g)$, then $g(\nabla_{\dot{\Gamma}} \dot{\Gamma}, N) = \kappa_{\Gamma} = \dot{\sigma}$. In summary, given $H \in \mathbb{R}$, the curve $\Gamma = \Gamma(s)$ generates after the action of $SO(n-1)$ an H -hypersurface of $(\overline{\mathbb{D}}^* \times \mathbb{R}, g)$ if and only if $r(s), z(s), \sigma(s)$ satisfy the ODE system

$$\begin{cases} \dot{r} &= \cos \sigma \\ \dot{z} &= \sin \sigma \\ \dot{\sigma} &= (n-1)H - (n-2)\frac{\phi'(r)}{\phi(r)} \sin \sigma. \end{cases} \quad (5)$$

The above system only makes sense in \mathcal{B}^* . Nevertheless, it is well-known that solutions Γ of (5) that go to the inner boundary $\{r = 0\}$ of \mathcal{B}^* must enter perpendicularly (see e.g., Eells and Ratto [3] or Hsiang and Hsiang [7]). Three other basic properties of the system (5) are the following ones.

- P1. Any vertical translation of a solution of (5) is also a solution of (5). In particular, (5) admits a first integral.
- P2. If a solution $\Gamma(s)$ of (5) defined in $(s_0 - \varepsilon, s_0]$ has vertical tangent line at s_0 , i.e., $\sigma(s_0) = \pm\pi/2$, then $\Gamma(s)$ can be extended by reflecting across the horizontal line $\{z = z(s_0)\}$ to a solution of (5) defined in the interval $(s_0 - \varepsilon, s_0 + \varepsilon)$.
- P3. The vertical line $\Gamma(s) = (r, s)$ (with $r \in (0, 1]$ fixed) is a solution of (5), and the value H of the mean curvature of the corresponding vertical cylinder $C(r) = (\partial\mathbb{D}(r)) \times \mathbb{R}$ is $H = \frac{n-2}{n-1} \frac{\phi'(r)}{\phi(r)}$.

Consider the smooth function $h: [0, 1] \rightarrow \mathbb{R}$ given by $h(r) = (n-1)H \int_0^r \phi(u)^{n-2} du$. It is straightforward to show that

$$J(s) = \phi(r(s))^{n-2} \sin \sigma(s) - h(r(s)) \quad (6)$$

is a first integral of (5), i.e., $J(s)$ is constant along any solution of (5).

Remark 4.3 It is worth mentioning the following geometric interpretation of the first integral J . The Killing vector field $\partial_z = \frac{\partial}{\partial z}$ on $(\overline{\mathbb{D}} \times \mathbb{R}, g)$ induces a notion of *scalar flux* on every H -hypersurface Σ (not necessarily of revolution), defined as

$$\text{Flux}(\Sigma, \gamma) = \int_{\gamma} \langle \partial_z, \eta \rangle + (n-1)H \int_D \langle \partial_z, N_D \rangle,$$

where $\gamma \subset \Sigma$ is a $(n-2)$ -cycle on Σ , $D \subset \overline{\mathbb{D}} \times \mathbb{R}$ is a compact hypersurface with $\partial D = \gamma = D \cap \Sigma$ and N_D is a unit normal vector field to D . The divergence theorem shows that $\text{Flux}(\Sigma, \gamma)$ only depends on the homology class of γ in Σ . In the particular case that Σ is a hypersurface of revolution and γ is the $SO(n-1)$ -orbit of a point in Σ , then $\text{Flux}(\Sigma, \gamma) = J$. This equality explains why $J = 0$ is the value of the first integral for CMC spheres (since γ is null homologous in this case) in the particular cases $n = 3$, $\phi(r) = r$ (which produces the flat standard metric on $\overline{\mathbb{D}}^2$), $\phi(r) = \sin(r)$ (so $(\overline{\mathbb{D}}, ds^2)$ is isometric to a closed hemisphere in $\mathbb{S}^2(1)$) and $\phi(r) = \sinh(r)$ (and $(\overline{\mathbb{D}}, ds^2)$ can be viewed inside $\mathbb{H}^2(-1)$).

Rather than classifying all solutions of (5) by analyzing every possible value of J , we will only study the case $J = 0$ as we are interested in producing an example of an H -hypersurface of $\mathbb{D} \times \mathbb{R}$ that resembles in a certain sense, a graphical CMC halfsphere; by this we mean the following. We will choose ϕ so that the behavior of the profile curve Γ solution of (5) is similar to the one of a lower half-sphere, but with infinite length and asymptotic to a vertical line instead of achieving the vertical direction for its tangent line in a compact portion of curve.

Suppose that $\Gamma(s)$ is a solution of (5) with $J = 0$. Thus, $\sin \sigma = h/\phi^{n-2}$ and $\cos \sigma = \sqrt{1 - (h/\phi^{n-2})^2}$. These equations only make sense if h/ϕ^{n-2} takes values in $(-1, 1)$. With this observation in mind, it is easy to deduce from the equation $h' = (n-1)H\phi^{n-2}$ and L'Hôpital's rule that

$$\lim_{r \rightarrow 0^+} \frac{h(r)}{\phi(r)^{n-2}} = 0, \quad \lim_{r \rightarrow 0^+} \left(\frac{h}{\phi^{n-2}} \right)'(r) = H. \quad (7)$$

This implies that the number

$$r_1 = \sup \left\{ r \in (0, 1] : \frac{h(r)}{\phi(r)^{n-2}} \in (0, 1) \text{ for all } r \in (0, r_1) \right\}$$

exists (it clearly depends on the choice of ϕ).

Note that if $r_1 < 1$, then $\frac{h(r_1)}{\phi(r_1)^{n-2}} = 1$; otherwise $r_1 = 1$ and we only can ensure that $\lim_{r \rightarrow 1^-} \frac{h(r)}{\phi(r)^{n-2}} \leq 1$. Therefore, in $(0, r_1)$ we have

$$\frac{dz}{dr} = \frac{dz}{ds} \frac{ds}{dr} = \frac{\sin \sigma}{\cos \sigma} = \frac{h/\phi^{n-2}}{\sqrt{1 - (h/\phi^{n-2})^2}}. \quad (8)$$

From (7) and (8) we deduce that $z = z(r)$ is an increasing function of $r \in [0, r_1)$ (strictly increasing if $r > 0$), whose graph Γ intersects the vertical line $\{r = 0\}$ orthogonally.

We next compute the length of Γ :

$$\text{length}(\Gamma)_0^R = \int_0^R \sqrt{1 + \left(\frac{dz}{dr} \right)^2} dr \stackrel{(8)}{=} \int_0^R \frac{dr}{\sqrt{1 - (h/\phi^{n-2})^2}}, \quad (9)$$

and we want to choose ϕ so that $\lim_{R \rightarrow r_1^-} \text{length}(\Gamma)_0^R = +\infty$. Note that if we impose

$$\frac{h}{\phi^{n-2}} = \sqrt{1 - (r_1 - r)^2} \quad \text{in } [r'_1, r_1), \quad (10)$$

for some $r'_1 \in (0, r_1)$, then

$$\text{length}(\Gamma)_{r'_1}^R \stackrel{(9)}{=} \int_{r'_1}^R \frac{dr}{r_1 - r} = \log \left(\frac{r_1 - r'_1}{r_1 - R} \right), \quad (11)$$

which limits to $+\infty$ if $R \rightarrow r_1^-$, as desired. Equations (10) and $h' = (n-1)H\phi^{n-2}$ lead to an ODE of order one for h whose solution is $h(r) = e^{(n-1)H \arcsin(r-r_1)}$. This function h produces

$$\phi: [0, r_1) \rightarrow \mathbb{R}, \quad \phi(r) = \frac{e^{\frac{n-1}{n-2}H \arcsin(r-r_1)}}{[1 - (r_1 - r)^2]^{\frac{1}{2(n-2)}}}. \quad (12)$$

We therefore conclude that with the choice of ϕ given by (12) for $r_1 \in (0, 1)$ fixed but arbitrary, there exists a solution Γ of (5) with first integral $J = 0$, that can be parameterized by $\Gamma(r) = (r, z(r))$ with $z: (0, r_1) \rightarrow \mathbb{R}$ a smooth function that satisfies $z'(r) > 0$ for all $r \in (0, 1)$ (by (8) and (10)) and $z(r) \rightarrow +\infty$ as $r \rightarrow r_1^-$ (by (11)).

Note that with definition (12), ϕ fails to satisfy the conditions to define a smooth metric at $r = 0$ (for instance, $\phi(0) > 0$). This problem can be overcome by truncating ϕ appropriately; to do this, fix $r_0 \in (0, r_1)$ and substitute the definition of ϕ in (12) in $[0, r_0]$ by a smooth positive function so that the resulting function, also called ϕ , is of class C^∞ at $r = r_0$ and $\phi'(0) = 1$, $\phi^{(2k)}(0) = 0$ for all $k \in \mathbb{N} \cup \{0\}$, and $h(r)/\phi(r)^{n-2}$ keeps taking values in $[0, 1)$ for all $r \in [0, r_1)$. Plugging this new function $\phi: [0, r_1) \rightarrow (0, \infty)$ in (2) we define a smooth, $SO(n-1)$ -invariant metric in $\mathbb{D}(r_1)$ (not analytic). By uniqueness of solutions of the ODE system (5) with given initial values, we also conclude that restriction $\Gamma|_{[r_0, r_1)}$ of the graphical solution Γ appearing in the last paragraph can be smoothly extended to $[0, r_1)$ as a solution of (5) with the new function ϕ . The resulting solution, relabeled as $\Gamma(r) = (r, z(r))$, is again a vertical graph over $[0, r_1)$ with $z'(0) = 0$, $z(r)$ strictly increasing in $(0, r_1)$ and $z(r) \rightarrow +\infty$ as $r \rightarrow r_1^-$.

We now extend ϕ to $[r_1, 1]$. Note that (12) defines a smooth function at $r = r_1$, so simply extend ϕ smoothly to $[0, 1]$ by a positive function in $[r_1, 1]$, also denoted by ϕ , all whose derivatives coincide with the corresponding ones of (12) at r_1 . Plugging this function ϕ in (2) we obtain a smooth, $SO(n-1)$ -invariant metric ds^2 on $\overline{\mathbb{D}}$. Recall that for any choice of ϕ , the vertical cylinder $C(r)$ of radius r in $(\overline{\mathbb{D}} \times \mathbb{R}, g)$ has constant mean curvature $H = \frac{n-2}{n-1} \frac{\phi'(r)}{\phi(r)}$. As the graphical solution $\Gamma(r) = (r, z(r))$, $r \in [0, r_1)$, constructed in the last paragraph generates an $SO(n-1)$ -invariant H -hypersurface Σ which is smoothly asymptotic to the upper end of the vertical cylinder $C(r_1)$, then the mean curvature of $C(r_1)$ is also H , so this must be the value of $\frac{n-2}{n-1} \frac{\phi'(r)}{\phi(r)}$ (this can be checked by direct computation in (12)). Finally, we define the desired CMC foliation \mathcal{F} as the vertical translates of Σ in $\mathbb{D}(r_1) \times \mathbb{R}$ together with the collection of vertical cylinders $C(r)$, $r \in [r_1, 1]$.

Remark 4.4 (A) Let $r_2 \in (r_1, 1)$. If we choose the extension of ϕ to $[r_1, 1]$ so that it additionally becomes constant in $[r_2, 1]$, then the ambient metric on $(\overline{\mathbb{D}}(1) - \mathbb{D}(r_2)) \times \mathbb{R}$ is flat and the mean curvatures of the cylinders $C(r)$, $r \in [r_2, 1]$ are zero. Note that in the above construction, $H > 0$, and $0 < r_1 < r_2$ are arbitrary.

(B) The construction of the CMC foliation \mathcal{F} of $\overline{\mathbb{D}}(1) \times \mathbb{R}$ induces a CMC foliation of $\overline{\mathbb{D}}(1) \times (\mathbb{R}/\lambda\mathbb{Z})$ for any value of $\lambda > 0$. Observe that the $(n-1)$ -dimensional volume of the compact hypersurface $(\partial\mathbb{D}(\delta)) \times (\mathbb{R}/\lambda\mathbb{Z})$ is independent of $\delta \in [r_2, 1]$ and of $H > 0$, and it can be prescribed arbitrarily by picking the appropriate value of λ .

Definition 4.5 We say that a codimension-one smooth foliation \mathcal{F} of a smooth n -manifold X contains an *enlarged Reeb component* $\Omega \subset X$ if Ω is a smooth saturated domain and there exists a diffeomorphism $\phi: \overline{\mathbb{D}} \times \mathbb{S}^1 \rightarrow \overline{\Omega}$ such that the pullback foliation $\phi^*(\mathcal{F}|_{\overline{\Omega}})$ is an enlarged Reeb-type foliation of $\overline{\mathbb{D}} \times \mathbb{S}^1$, according to Definition 4.2. We refer to the image foliated region $\phi(\overline{\mathbb{D}} \times \mathbb{S}^1) \subset X$ as an *enlarged Reeb component of \mathcal{F}* . Note that every enlarged Reeb component of \mathcal{F} contains a classical Reeb component, according to Definition 4.1.

5 The proof of Theorem 1.1.

Let X be a closed smooth n -manifold. The existence of a smooth, transversely oriented codimension-one foliation of X implies that the Euler characteristic of X vanishes (apply the Poincaré-Hopf index theorem to the unit normal vector field to the transversely oriented foliation with respect to an arbitrarily chosen metric on X). Therefore the necessary implication in Theorem 1.1 is clear. In fact, item (a) of Theorem 1 in Thurston [20] shows that X admits a smooth foliation if and only if it has vanishing Euler characteristic. Since such an X with Euler characteristic zero always admits a nowhere zero smooth vector field V (Hopf [6], also see Theorem 39.7 in Steenrod [18]), then item (b) of Theorem 1 in [20] assures that the transversely oriented $(n - 1)$ -plane field on X given by the orthogonal complement of V with respect to a previously chosen metric on X , is homotopic to the tangent plane of a smooth, codimension-one foliation. In particular, there exists a smooth, codimension-one, transversely oriented foliation on X .

Henceforth, to prove the sufficient implication of Theorem 1.1 we can assume that X admits a smooth, codimension-one, transversely oriented foliation \mathcal{F} . We will also fix an auxiliary Riemannian metric g on X . Our goal will be to make a possibly different choice of \mathcal{F} and g so that \mathcal{F} is a non-minimal CMC foliation with respect to g .

5.1 Turbularization along an embedded closed transversal.

Consider \mathbb{S}^1 to be the quotient \mathbb{R}/\mathbb{Z} with the orientation induced by the usual orientation on \mathbb{R} and let $\Gamma: \mathbb{S}^1 \rightarrow X$ be an embedded smooth curve transverse to \mathcal{F} , which exists by the following elementary argument. Consider a maximal integral curve γ of the unit normal field $N_{\mathcal{F}}$ to \mathcal{F} . If γ is closed, then we are done. If γ never closes, then the compactness of X implies that γ enters twice (actually, infinitely many times) inside some product coordinate chart $U = \mathbb{D} \times (0, 1)$ of the foliation. Thus, $(X - U) \cap \gamma$ contains a subarc $\hat{\gamma}$, one of whose end points $\hat{\gamma}(0)$ lies in $\mathbb{D} \times \{0\}$ and the other one $\hat{\gamma}(1)$ lies in $\mathbb{D} \times \{1\}$. By basically joining $\hat{\gamma}(0)$ with $\hat{\gamma}(1)$ by a ‘straight line segment’ in U and then smoothing the resulting embedded closed curve, we construct the desired embedded closed transversal $\Gamma: \mathbb{S}^1 \rightarrow X$ to \mathcal{F} . Finally, after replacing Γ by a small perturbation of its two-sheeted cover, we will assume that a small regular neighborhood of Γ is orientable. We can also suppose that the inner product with respect to g of the velocity vector field to Γ with $N_{\mathcal{F}}$ is positive.

Given $\sigma \in \{+, -\}$, Γ and \mathcal{F} , we next describe a method for modifying \mathcal{F} in a small regular neighborhood of Γ giving rise to a new foliation $\mathcal{F}(\Gamma, \sigma)$ of X , by means of a standard technique called *turbularization*. Later we will introduce some metric aspects in the turbularization process that will be useful in our goal to prove the necessary implication of Theorem 1.1.

First choose $\varepsilon > 0$ sufficiently small so that the closed embedded 8ε -neighborhood $V(\Gamma, 8\varepsilon)$ of Γ is parameterized by a diffeomorphism

$$\Phi: \overline{\mathbb{D}}(8\varepsilon) \times \mathbb{S}^1 \rightarrow V(\Gamma, 8\varepsilon), \quad (13)$$

where $\overline{\mathbb{D}}(r) = \{x \in \mathbb{R}^{n-1} \mid \|x\| \leq r\}$ for each $r > 0$. We can also take Φ so that the restricted foliation $\mathcal{F}|_{V(\Gamma, 8\varepsilon)}$ of $V(\Gamma, 8\varepsilon)$ consists of $\{\Phi(\overline{\mathbb{D}}(8\varepsilon) \times \{\theta\}) \mid \theta \in \mathbb{S}^1\}$; in

particular, all of these leaves are $(n-1)$ -dimensional disks. Note that each orbit of the action of \mathbb{S}^1 on $V(\Gamma, 8\varepsilon)$, induced by pushing forward via Φ the product action of \mathbb{S}^1 on $\overline{\mathbb{D}}(8\varepsilon) \times \mathbb{S}^1$, intersects each leaf of $\mathcal{F}|_{V(\Gamma, 8\varepsilon)}$ transversely in a single point. In particular, given $r \in (0, 8\varepsilon]$, the compact hypersurface

$$\mathbb{T}(r) = \Phi(\partial\overline{\mathbb{D}}(r) \times \mathbb{S}^1)$$

obtained as the orbit of the action of \mathbb{S}^1 on $\partial\overline{\mathbb{D}}(r)$, intersects each of the disk-type leaves of $\mathcal{F}|_{V(\Gamma, 8\varepsilon)}$ transversely in an embedded $(n-2)$ -sphere. $SO(n-1) \times \mathbb{S}^1$ acts naturally on $\overline{\mathbb{D}}(8\varepsilon) \times \mathbb{S}^1$ (hence on $V(\Gamma, 8\varepsilon)$ via Φ) so that each $(A, \theta) \in SO(n-1) \times \mathbb{S}^1$ acts by rotation by A in $\overline{\mathbb{D}}(\varepsilon)$ and by translation by θ in \mathbb{S}^1 .

Given $\sigma \in \{+, -\}$, let S_σ be a connected, complete, non-compact smooth hypersurface with compact boundary in $V(\Gamma, 8\varepsilon)$ with the following properties:

- S1. S_σ is contained in $\Phi[(\overline{\mathbb{D}}(8\varepsilon) - \overline{\mathbb{D}}(4\varepsilon)) \times \mathbb{S}^1]$ and it is of revolution, i.e., if $\Phi(x, \theta) \in S_\sigma$, then $\Phi(Ax, \theta) \in S_\sigma$ for any $A \in SO(n-1)$.
- S2. S_σ is graphical (with respect to the orbits of the \mathbb{S}^1 -action on the second factor) over the annulus $\Phi[(\mathbb{D}(6\varepsilon) - \overline{\mathbb{D}}(4\varepsilon)) \times \mathbb{S}^1]$.
- S3. The intersection of S_σ with $\Phi[(\overline{\mathbb{D}}(8\varepsilon) - \overline{\mathbb{D}}(6\varepsilon)) \times \mathbb{S}^1]$ equals the annulus $\Phi[(\overline{\mathbb{D}}(8\varepsilon) - \overline{\mathbb{D}}(6\varepsilon)) \times \{0\}]$ (recall that this annulus is part of a leaf of $\mathcal{F}|_{V(\Gamma, 8\varepsilon)}$).
- S4. S_σ is smoothly asymptotic in $V(\Gamma, 8\varepsilon)$ to the hypersurface $\mathbb{T}(4\varepsilon)$.
- S5. The orientation of $\mathbb{T}(4\varepsilon)$ induced by the annular graph S_σ coincides with the inward pointing unit normal to the solid region $\Phi(\mathbb{D}(4\varepsilon) \times \mathbb{S}^1)$ when $\sigma = +$, and with the outward pointing unit normal when $\sigma = -$; see Fig. 5.1 for the case of $S_\sigma = S_+$.

We are now in a position to describe $\mathcal{F}(\Gamma, \sigma)$. Note that Property S2 above implies that the family of translates $S_\sigma(\theta)$ of S_σ by elements $\theta \in \mathbb{S}^1$ in the second factor of the $(SO(n-1) \times \mathbb{S}^1)$ -action on $V(\Gamma, 8\varepsilon)$, are all disjoint. These translates $S_\sigma(\theta)$, $\theta \in \mathbb{S}^1$, together with the hypersurfaces $\mathbb{T}(t)$, $t \in (2\varepsilon, 4\varepsilon]$ and with the leaves of $\mathcal{F} \cap [X - \Phi(\mathbb{D}(6\varepsilon) \times \mathbb{S}^1)]$ produce a smooth, transversely oriented foliation of $X - \Phi(\overline{\mathbb{D}}(2\varepsilon) \times \mathbb{S}^1)$. In turn, this foliation can be extended to a smooth, transversely oriented foliation $\mathcal{F}(\Gamma, \sigma)$ of X by attaching an enlarged Reeb component $\Omega = \Omega(\Gamma, \sigma)$ along the boundary $\mathbb{T}(2\varepsilon)$ of $X - \Phi(\overline{\mathbb{D}}(2\varepsilon) \times \mathbb{S}^1)$ and so that the \mathbb{S}^1 -action on $\Phi[(\overline{\mathbb{D}}(8\varepsilon) - \overline{\mathbb{D}}(2\varepsilon)) \times \mathbb{S}^1]$ agrees with the natural translational action of \mathbb{S}^1 on Ω .

5.2 Constructing the desired foliation of X .

Using the above construction of $\mathcal{F}(\Gamma, \sigma)$ from \mathcal{F} , Γ and σ , we will show how to obtain a smooth, transversely oriented, codimension-one foliation of X such that with respect to some Riemannian metric on X (to be defined in Sections 5.3 and 5.4), has leaves of constant mean curvature. We start with a smooth, transversely oriented, codimension-one foliation \mathcal{F} on X . After possibly doing tubularization along a closed transversal to \mathcal{F} (such a closed transversal was proven to exist at the beginning of Section 5.1), we can assume that \mathcal{F} contains a non-compact leaf.

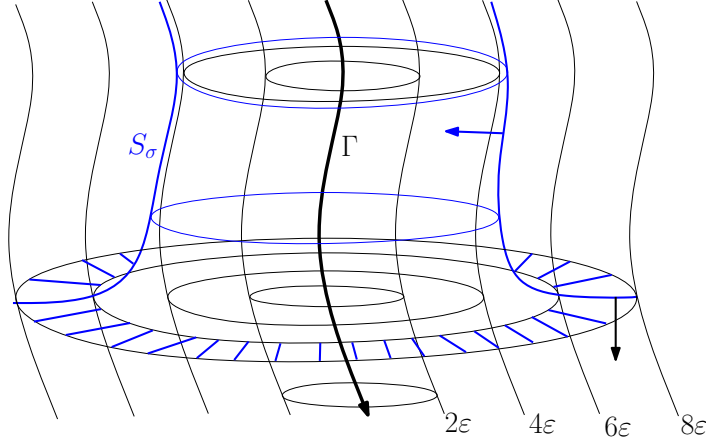


Figure 4: The complete non-compact hypersurface $S_\sigma = S_+$ in an 8ϵ -regular neighborhood of a closed transversal Γ . Note that as Γ is closed, then S_σ comes again into the represented portion of X , lying in between the represented portion of itself and the compact hypersurface $\mathbb{T}(4\epsilon)$, and this happens infinitely many times as S_σ wraps infinitely often around $\mathbb{T}(4\epsilon)$ limiting to this compact hypersurface.

Lemma 5.1 *There exists a finite collection $\Delta = \{\gamma_1, \dots, \gamma_k\}$ of pairwise disjoint, compact embedded arcs that are transverse to the leaves of \mathcal{F} and positively oriented with respect to $N_{\mathcal{F}}$, such that:*

1. *Every compact leaf of \mathcal{F} intersects at least one of the γ_i .*
2. *For each $i = 1, \dots, k$, the end points of γ_i lie on non-compact leaves of \mathcal{F} .*

Proof. Let \mathcal{F}_0 be the set of compact leaves of \mathcal{F} . After picking a metric on X , we can endow \mathcal{F}_0 with the structure of a compact metric space with the induced distance between compact leaves (compactness of \mathcal{F}_0 follows from Haefliger [5], also see Theorem 6.1.1 in [1]). Given a leaf $L \in \mathcal{F}_0$, then either L lies in a maximal, compact oriented 1-parameter family $I = \{L_t \mid t \in [0, 1]\} \subset \mathcal{F}_0$ or it fails to have this property.

In the first case, the leaves L_0, L_1 are limits of non-compact leaves of \mathcal{F} . Pick points $p(0), p(1)$ in non-compact leaves of \mathcal{F} sufficiently close to L_0, L_1 so that there exists a positively oriented transversal arc γ_L joining $p(0)$ with $p(1)$ and which intersects exactly once each of the leaves in I . If on the contrary, L does not lie in any compact oriented 1-parameter family I as before, then a slight modification of the above arguments applies to L in order to find an arbitrarily short, positively oriented transversal arc γ_L intersecting L exactly once and with end points $p(0), p(1)$ in non-compact leaves of \mathcal{F} .

Hence we have associated to each $L \in \mathcal{F}_0$ an open arc γ_L that intersects L with end points in $X - \mathcal{C}_{\mathcal{F}}$, where $\mathcal{C}_{\mathcal{F}} = \cup_{L \in \mathcal{F}_0} L$. For each open transversal arc γ_L as before, consider the set of leaves $A(\gamma_L)$ in \mathcal{F}_0 that intersect γ_L . Clearly $A(\gamma_L)$ is an open set in \mathcal{F}_0 . As \mathcal{F}_0 is compact, then we can extract a finite open subcover from the family $\{A(\gamma_L) \mid L \in \mathcal{F}_0\}$, and the lemma follows by choosing the transversal arcs associated to the finite subcover. \square

Our next goal is to modify \mathcal{F} by turbularization along pairs Γ_1^i, Γ_2^i of disjoint closed transversals associated to each $\gamma_i \in \Delta$ with the notation of Lemma 5.1. We next explain how to associate such a pair Γ_1^i, Γ_2^i to γ_i . Consider one of the oriented arcs $\gamma \in \Delta$ appearing in Lemma 5.1. Note that γ intersects $\mathcal{C}_{\mathcal{F}}$ in a compact set. As we travel along γ with its orientation, there exists a first point $p(\gamma)$ in γ that lies in a compact leaf of \mathcal{F} , and a last point $q(\gamma)$ in γ that lies in a compact leaf of \mathcal{F} . We label by $L(p(\gamma)), L(q(\gamma))$ the (compact) leaves of \mathcal{F} passing through $p(\gamma)$ and $q(\gamma)$, respectively. Note that

- $L(p(\gamma))$ has non-trivial holonomy on its side containing the starting point of γ .
- $L(q(\gamma))$ has non-trivial holonomy on its side containing the end point of γ .

A brief explanation of the above expression 'non-trivial holonomy' is in order. We will explain this property at $L = L(p(\gamma))$; the case at $L(q(\gamma))$ is similar. Suppose that $\varphi_\beta: \mathbb{D} \times (0, 1) \rightarrow U_\beta$ is a foliation chart (we use the notation in Definition 2.1), so that $\varphi_\beta(\mathbb{D} \times \{1/2\}) \subset L$ and $\varphi_\beta(\vec{0}, 1/2) = p(\gamma)$, where $\vec{0}$ denotes the origin in \mathbb{D} . The component γ_1 of $\gamma \cap U_\beta$ that passes through $p(\gamma)$ can be identified with $\{\vec{0}\} \times (0, 1)$. Given an embedded, smooth transversal arc $\alpha \subset U_\beta$ with end points $\alpha(0) \in \mathbb{D} \times \{0\}$, $\alpha(1) \in \mathbb{D} \times \{1\}$, the local product structure of the foliation chart determines a diffeomorphism $h: \gamma_1 \rightarrow \alpha$ so that

$$\varphi_\beta^{-1} \left(h(\varphi_\beta(\vec{0}, t)) \right) \in \mathbb{D} \times \{t\}, \quad \text{for all } t \in (0, 1). \quad (14)$$

If $c: [0, 1] \rightarrow L$ is a path starting at $p(\gamma)$, then we can cover $c([0, 1])$ by a finite number of foliation charts $(U_{\beta_1}, \varphi_{\beta_1}), \dots, (U_{\beta_k}, \varphi_{\beta_k})$ as before, so that $U_{\beta_i} \cap U_{\beta_{i+1}} \neq \emptyset$ for all i . Pick embedded, smooth transversal arcs

$$\alpha_1 = \gamma_1 \subset U_1, \alpha_2 \subset U_{\beta_1} \cap U_{\beta_2}, \dots, \alpha_{k-1} \subset U_{\beta_{k-1}} \cap U_{\beta_k}, \alpha_k \subset U_{\beta_k}$$

with $\alpha_i(1/2) \in c([0, 1])$ for all i , $\alpha_k(1/2) = c(1)$ and end points $\alpha_i(0) \in \mathbb{D} \times \{0\}$, $\alpha_i(1) \in \mathbb{D} \times \{1\}$. Then, (14) produces diffeomorphisms $h_i: \alpha_i \rightarrow \alpha_{i+1}$, $1 = 1, \dots, k-1$. The composition $h_{k-1} \circ \dots \circ h_1$ is a diffeomorphism from the germ of α_1 at $c(0) = p(\gamma)$ to the germ of α_k at $c(1)$, called *holonomy transport*. It is a well-known fact that the holonomy transport only depends on the relative homotopy of c . In the particular case that c is a loop based at $p(\gamma)$, the homotopy invariance of the holonomy transport gives rise to a homomorphism

$$\text{Hol}: \pi_1(L, p(\gamma)) \rightarrow G(\gamma)$$

from the fundamental group of L based at $p(\gamma)$ to the group of germs of diffeomorphisms of the transversal arc γ at $p(\gamma)$. The 'non-trivial holonomy' mentioned above means that the homomorphism Hol is not constant equal to the identity element in $G(\gamma)$.

As $p(\gamma)$ is the first point in γ that lies in a compact leaf of \mathcal{F} , then there exists a closed embedded curve $c_1 \subset L(p(\gamma))$ whose homotopy class $[c_1] \in \pi_1(L, p(\gamma))$ produces a nontrivial diffeomorphism $\text{Hol}([c_1]) \in G(\gamma)$; more specifically, there exists a left normal fence above c_1 and non-compact leaves of \mathcal{F} limiting to $L(p(\gamma))$ on the local side of $L(p(\gamma))$ that contains the initial point of γ . In fact, after a small perturbation of γ we may assume that this normal fence is chosen to contain the subarc $\gamma|_{p(\gamma)}^0$ of γ between

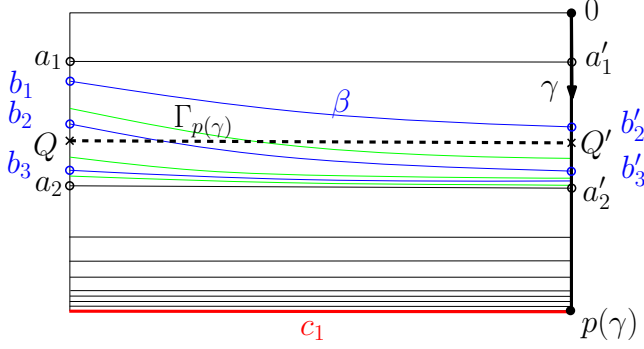


Figure 5: A left normal annular fence A above c_1 (identify the two “vertical sides” of A by horizontal translation) transversely intersects infinitely many leaves of \mathcal{F} different from $L(p(\gamma))$, all of them non-compact. The non-trivial holonomy of c_1 in A implies that at least one of the intersection curves of A with leaves of \mathcal{F} is a non-compact embedded spiraling curve β (the points $b_j, b'_j \in \beta$ identify for each $j \in \mathbb{N}$) that limits in A to some closed intersection curve (with end points a_2, a'_2 in the picture) which could be c_1 . Spiraling intersection curves near c_1 are then grouped into open annular components in A , each of which is bounded by closed intersection curves; the closed transversal $\Gamma_{p(\gamma)}$ can be then constructed by transversely crossing every spiraling intersection curve in a given annular component; note that the two end points of the above representation of $\Gamma_{p(\gamma)}$ identify to a single point $Q = Q'$.

the beginning point of γ and $p(\gamma)$ (after possibly shortening γ). A standard argument allows us to find a positively oriented closed transversal $\Gamma_{p(\gamma)}$ to \mathcal{F} that lies in the left normal fence, intersects $\gamma|_{p(\gamma)}^0$ at a single point and is topologically parallel to c_1 , see Fig. 5.

Let $\mathcal{F}(\Gamma_{p(\gamma)}, -)$ be the smooth foliation of X constructed by turbularization around $\Gamma_{p(\gamma)}$ as we explained in Section 5.1; in particular, $\mathcal{F}(\Gamma_{p(\gamma)}, -)$ contains an enlarged Reeb component $\Omega(\Gamma_{p(\gamma)}, -) \subset \Phi(\overline{\mathbb{D}}(3\varepsilon) \times \mathbb{S}^1)$ (here we are using the notation in (13)) and $\mathcal{F}(\Gamma_{p(\gamma)}, -)$ coincides with the previous foliation \mathcal{F} outside $\Phi(\overline{\mathbb{D}}(6\varepsilon) \times \mathbb{S}^1)$.

We next do a similar construction on the “future” side of $q(\gamma)$, i.e., there exists a closed embedded curve $c_2 \subset L(q(\gamma))$, a right normal fence above c_2 containing the subarc $\gamma|_1^{q(\gamma)}$ of γ between $q(\gamma)$ and the ending point of γ , and a positively oriented closed transversal $\Gamma_{q(\gamma)}$ to \mathcal{F} that lies in the normal fence, intersects $\gamma|_1^{q(\gamma)}$ at a single point and is topologically parallel to c_2 , and then we do turbularization around $\Gamma_{q(\gamma)}$ by constructing a new smooth foliation $\mathcal{F}(\Gamma_{q(\gamma)}, +)$ of X that contains an enlarged Reeb component $\Omega(\Gamma_{q(\gamma)}, +)$ and such that $\mathcal{F}(\Gamma_{q(\gamma)}, +)$ coincides with the previous foliation $\mathcal{F}(\Gamma_{p(\gamma)}, -)$ outside an embedded neighborhood of $\Gamma_{q(\gamma)}$. Note that $\mathcal{F}(\Gamma_{q(\gamma)}, +)$ also contains the enlarged Reeb component $\Omega(\Gamma_{p(\gamma)}, -)$ and that both enlarged Reeb components $\Omega(\Gamma_{p(\gamma)}, -)$, $\Omega(\Gamma_{q(\gamma)}, +)$ can be assumed to be disjoint.

Finally, we repeat the above process for each transversal arc $\gamma_i \in \Delta$ appearing in Lemma 5.1, increasing the number of pairwise disjoint enlarged Reeb components (two for each γ_i) until producing a smooth foliation \mathcal{F}' that contains $2k$ enlarged Reeb components $\Omega(\Gamma_{p(\gamma_i)}, -) \subset \Phi_2^i(\overline{\mathbb{D}}(3\varepsilon) \times \mathbb{S}^1)$, $\Omega(\Gamma_{q(\gamma_i)}, +) \subset \Phi_1^i(\overline{\mathbb{D}}(3\varepsilon) \times \mathbb{S}^1)$, $i = 1, \dots, k$,

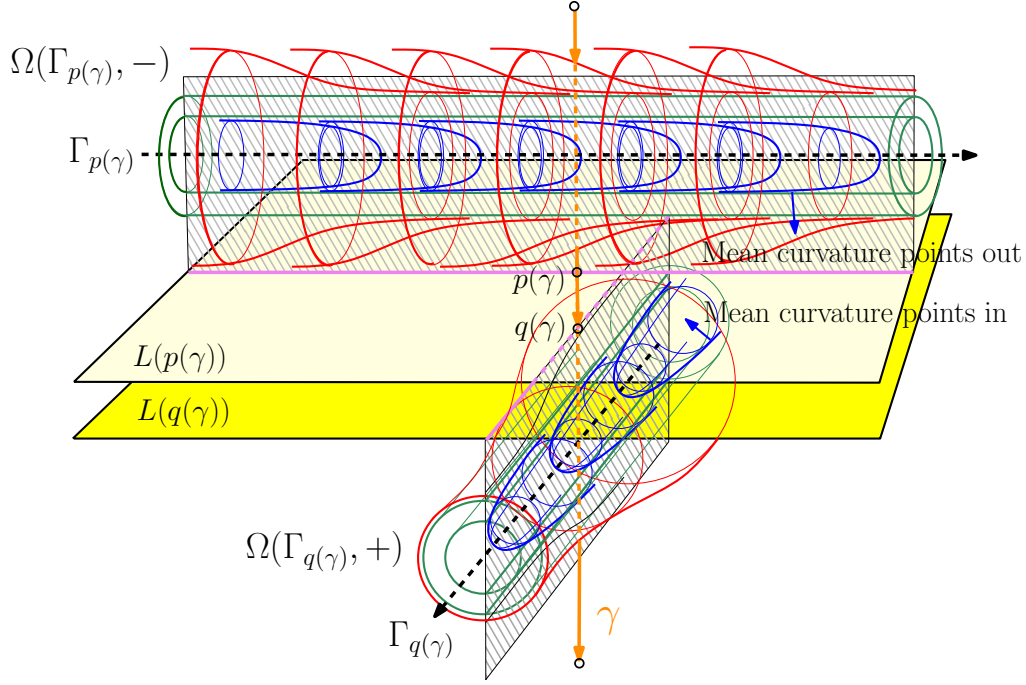


Figure 6: The enlarged Reeb components $\Omega(\Gamma_{q(\gamma)}, +)$, $\Omega(\Gamma_{p(\gamma)}, -)$ crossing each of the open transversal arcs $\gamma \in \Delta$.

where $\Gamma_1^i = \Gamma_{q(\gamma)}$, $\Gamma_2^i = \Gamma_{p(\gamma)}$ for each open transversal arc $\gamma = \gamma_i \in \Delta$, and the $\Phi_j^i: \overline{\mathbb{D}}(8\varepsilon) \times \mathbb{S}^1 \rightarrow V(\Gamma_j^i, 8\varepsilon)$ parameterize embedded tubular neighborhoods of the closed transversals Γ_j^i (clearly we can assume that $\varepsilon > 0$ is common to all these tubular neighborhoods), see Fig. 6.

5.3 Producing the desired metric on X : the case X is orientable.

The final step in our construction is to produce a smooth Riemannian metric on X that makes \mathcal{F}' a CMC foliation. To do so we will first analyze the case that X is orientable, in which the argument is simpler and based on Oshikiri's condition (Theorem 2.5). In next section we will give a different proof based on Sullivan's arguments in [19] which does not require either orientability for X or Oshikiri's result.

Consider the smooth function f on X which equals the mean curvature function of the enlarged Reeb foliations $\Omega(\Gamma_{p(\gamma)}, -)$, $\Omega(\Gamma_{q(\gamma)}, +)$ of \mathcal{F}' and is extended by zero to the complement of the union of these enlarged Reeb components. We now check that f satisfies the conditions of Theorem 2.5 with respect to the foliation \mathcal{F}' . First observe that, with the notation introduced at the end of Section 5.2, the portion $R(\Gamma_j^i, \pm)$ of each of the enlarged Reeb components $\Omega(\Gamma_j^i, \pm)$ given by the closure of their simply-connected leaves (this is, after unwrapping Γ_j^i , $R(\Gamma_j^i, \pm)$ lifts to $\overline{\mathbb{D}}(r_1) \times \mathbb{R}$ with the notation of item (E1) just before Definition 4.2), satisfies the following properties:

(R1) $R(\Gamma_j^i, \pm)$ is a Reeb component of \mathcal{F}' (see item 3 of Definition 4.1).

(R2) $R(\Gamma_j^i, \pm)$ is a minimal (+) or (-)-fcd.

(R3) f has points inside $R(\Gamma_j^i, \pm)$ with the same sign as the character of this Reeb component.

We now verify that there are no other minimal (+) or (-)-fcd in \mathcal{F}' . Consider a smooth, foliated compact domain D of \mathcal{F}' which is minimal under inclusion, and which is different from the above Reeb components $R(\Gamma_j^i, \pm)$ of \mathcal{F}' . We claim that D contains a boundary component where the transversal orientation points inward and another boundary component where the transversal orientation points outward (this would imply that there are no minimal (+) or (-)-fcd in \mathcal{F}' different from the Reeb components $R(\Gamma_j^i, \pm)$, which in turn implies that Oshikiri's condition holds for f and \mathcal{F}'). By Lemma 5.1, there exists a positively oriented transversal arc $\gamma_i \in \Delta$ that intersects transversely a boundary component ∂_1 of D at some point and by construction, γ_i contains a subarc s that enters the Reeb component $R(\Gamma_1^i, +)$ near the ending point of γ_i and exits its related "opposite" Reeb component $R(\Gamma_2^i, -)$ near the initial point of γ_i . As s also intersects ∂_1 at some point $s(t_0)$ and the end points of s lie outside of D , then $s \cap D$ must contain a further subarc s' with one of its boundary points being $s(t_0)$ and its other end point is $s(t_1)$ which lies in a second boundary component of D along which $N_{\mathcal{F}}$ is oppositely pointing with respect to D . Thus our claim is proved.

Finally, when X is orientable then Theorem 2.5 implies that there exists a smooth Riemannian metric g_1 on X whose mean curvature function is f , thereby proving the sufficient implication of Theorem 1.1 in this special case for X .

5.4 Producing the desired metric on X : the general case.

We next prove existence of a metric g_1 on X that makes \mathcal{F}' a CMC foliation, and which coincides up to scalings with the previously constructed smooth metrics from Section 4 in the finitely many enlarged Reeb components of \mathcal{F} of the form $\Omega(\Gamma_{p(\gamma)}, -)$, $\Omega(\Gamma_{q(\gamma)}, +)$ with the notation at the end of Section 5.2. This proof does not use Oshikiri's result and also works in the case X is non-orientable.

The key idea is, roughly speaking, to remove part of the enlarged Reeb components from \mathcal{F}' (so that a neighborhood of the resulting boundary is still a product foliation), glue appropriately the resulting boundary leaves in pairs to obtain a new compact n -manifold \hat{X} without boundary, where \mathcal{F}' induces a smooth codimension-one foliation $\hat{\mathcal{F}}$; then we will construct a smooth Riemannian metric \hat{g} on \hat{X} that makes $\hat{\mathcal{F}}$ minimal (by application of Sullivan's Theorem possibly generalized to the non-orientable case, which will be proved in Section 5.5). Finally, after pulling back \hat{g} to the complement of the Reeb components and gluing this pulled back metric with the ones constructed in Section 4 on the enlarged Reeb components (appropriately scaled by positive constants), we will obtain the desired smooth Riemannian metric. We now give details on this construction.

Recall that with the notation of Section 5.2, we have two closed, positively oriented transversals $\Gamma_1^i = \Gamma_{q(\gamma)}$, $\Gamma_2^i = \Gamma_{p(\gamma)}$ associated to each open transversal arc $\gamma = \gamma_i \in \Delta$. With the notation in Section 5.1, we also have related coordinates $\Phi_j^i(\overline{\mathbb{D}}(8\varepsilon) \times \mathbb{S}^1)$ for the replaced tubular neighborhoods of the Γ_j^i , $j = 1, 2$.

We consider the manifold \widehat{X} obtained as the quotient manifold

$$\widehat{X} = \left[X - \bigcup_{i=1}^k (\Phi_1^i(\mathbb{D}(3\varepsilon) \times \mathbb{S}^1) \cup \Phi_2^i(\mathbb{D}(3\varepsilon) \times \mathbb{S}^1)) \right] / \sim,$$

where \sim is the equivalence relation induced by the map $\Phi_1^i(x, \theta) \in \Phi_1^i(\partial\mathbb{D}(3\varepsilon) \times \mathbb{S}^1) \mapsto \Phi_2^i(x, -\theta) \in \partial\Phi_2^i(\mathbb{D}(3\varepsilon) \times \mathbb{S}^1)$. It is straightforward to check:

- (S1) \widehat{X} is orientable when X is orientable.
- (S2) The transversely oriented smooth foliation \mathcal{F}' on X induces a transversely oriented smooth foliation $\widehat{\mathcal{F}}$ of \widehat{X} .
- (S3) Every compact leaf of $\widehat{\mathcal{F}}$ is induced by a compact leaf of \mathcal{F}' and has a closed transversal passing through it (namely, the quotient by the equivalence relation \sim of a suitable subarc of one of the $\gamma_i \in \Delta$; note that for this to make sense, we possibly need to adjust the θ -variable in one of the two local ‘cylindrical’ coordinates (x, θ) defined by (13) so that the intersection of γ_i with $\Phi_1^i[\mathbb{D}(3\varepsilon) \times \mathbb{S}^1]$ and $\Phi_2^i[\mathbb{D}(3\varepsilon) \times \mathbb{S}^1]$ has coordinates $(x_0, 0)$ in both local systems).
- (S4) The foliation $\widehat{\mathcal{F}}$ is homologically taut, and hence Sullivan’s Theorem (or rather, its generalization Theorem 5.3 below) implies that \widehat{X} admits a smooth Riemannian metric $g_{\widehat{X}}$ such that all of the leaves of the foliation $\widehat{\mathcal{F}}$ are minimal.

We next consider the n -manifold with boundary

$$\widetilde{X} = X - \bigcup_{i=1}^k [\Phi_1^i(\mathbb{D}(3\varepsilon) \times \mathbb{S}^1) \cup \Phi_2^i(\mathbb{D}(3\varepsilon) \times \mathbb{S}^1)]. \quad (15)$$

We next describe how to glue the pulled back Riemannian metric \widetilde{g} and the pulled back minimal foliation $\widetilde{\mathcal{F}}$ on \widetilde{X} , obtained respectively from $g_{\widehat{X}}$, $\widehat{\mathcal{F}}$ on \widehat{X} , to the enlarged Reeb type components.

In the proof of the next assertion we will use some ideas of the proof of the main theorem in [19], more specifically, the paragraph in the proof of this result where Sullivan shows that *homological tautness implies geometrical tautness*. To do this, he first uses the homological tautness of an oriented foliation $\widehat{\mathcal{F}}$ of a compact manifold \widehat{X} , to produce via the Hahn-Banach theorem a closed $(n-1)$ -form ω on \widehat{X} whose restriction to the leaves of $\widehat{\mathcal{F}}$ is positive. Then he considers the pointwise linear map $P_\omega: T\widehat{X} \rightarrow T\widehat{\mathcal{F}}$ given by

$$P_\omega(v)_\perp(\omega|_{\widehat{F}}) = (v_\perp\omega)|_{\widehat{F}}, \quad v \in T_{\widehat{x}}\widehat{X}, \quad (16)$$

where $T\widehat{X}$ denotes that tangent bundle to \widehat{X} , $T\widehat{\mathcal{F}}$ is the subbundle of $T\widehat{X}$ tangent to the leaves of $\widehat{\mathcal{F}}$, \widehat{F} is the hyperplane tangent to $\widehat{\mathcal{F}}$ at a point $\widehat{x} \in \widehat{X}$ and \perp means contraction. In particular, $P_\omega(v) = v$ for all $v \in T\widehat{\mathcal{F}}$, hence P_ω is surjective. Next he defines the *purification* of ω as the $(n-1)$ -form on \widehat{X} given by $P_\omega^*(\omega|_{T\widehat{\mathcal{F}}})$ (in our codimension-one case, $P_\omega^*(\omega|_{T\widehat{\mathcal{F}}})$ coincides with ω). Finally, Sullivan constructs the

ambient smooth metric on \widehat{X} for which $\widehat{\mathcal{F}}$ is minimal by simply taking the orthogonal direct sum

$$g_{\widehat{X}} = g^1 \oplus g^2 \quad (17)$$

of any metric g^1 on the 1-dimensional distribution $\{\ker(P_\omega)\}$ with any metric g^2 on the subbundle $T\widehat{\mathcal{F}}$ whose $(n-1)$ -volume form coincides with $\omega|_{\widehat{\mathcal{F}}}$. In our proof of Assertion 5.2 below, we will choose the metrics g^1, g^2 appearing in (17) appropriately.

The next assertion uses the rotationally symmetric metric ds^2 defined by equation (2) for the function $\phi = \phi(r)$ given by (12) and extended to $r \in [0, 1]$ as explained in Remark 4.4 and in the paragraph just before this remark. Recall that by Remark 4.4, given $\lambda > 0$, the metric $ds^2 + dz^2$ on $\overline{\mathbb{D}}(1) \times (\mathbb{R}/\lambda\mathbb{Z})$ restricts to a product metric on $[\overline{\mathbb{D}}(1) - \mathbb{D}(r_2)] \times (\mathbb{R}/\lambda\mathbb{Z})$. In the sequel it will be useful to write this product Riemannian manifold in a different manner. Consider the diffeomorphism

$$\chi: [\overline{\mathbb{D}}(1) - \mathbb{D}(r_2)] \times (\mathbb{R}/\lambda\mathbb{Z}) \rightarrow (\mathbb{S}^{n-2}(r_2) \times (\mathbb{R}/\lambda\mathbb{Z})) \times [r_2, 1], \quad \chi(ra, b) = (r_2a, b, r),$$

where $a \in \mathbb{S}^{n-2}(1)$ and $r \in [r_2, 1]$. Denote by g_μ the product metric on $\mathbb{S}^{n-2}(r_2) \times (\mathbb{R}/\lambda\mathbb{Z})$ that makes χ an isometry from $ds^2 + dz^2$ to $g_\mu \times dr^2$. Note that the function ϕ in (12) depends on the choice of $H > 0$, but this dependence does not affect the metric \widetilde{g} defined just after (15).

Assertion 5.2 *There exist numbers $\lambda_1, \dots, \lambda_k > 0$ such that \widetilde{g} can be chosen so that the each of the domains $(\Phi_j^i([\overline{\mathbb{D}}(4\varepsilon) - \mathbb{D}(3\varepsilon)] \times \mathbb{S}^1), \widetilde{g})$ is isometric to the Riemannian product*

$$((\mathbb{S}^{n-2}(r_2) \times (\mathbb{R}/\lambda_i\mathbb{Z})) \times [r_2, 1], g_\mu \times dr^2),$$

by a diffeomorphism $\psi: \Phi_j^i([\overline{\mathbb{D}}(4\varepsilon) - \mathbb{D}(3\varepsilon)] \times \mathbb{S}^1) \rightarrow (\mathbb{S}^{n-2}(r_2) \times (\mathbb{R}/\lambda_i\mathbb{Z})) \times [r_2, 1]$ such that the pullback by ψ of the product foliation $\{[\mathbb{S}^{n-2}(r_2) \times (\mathbb{R}/\lambda_i\mathbb{Z})] \times \{\delta\} \mid \delta \in [r_2, 1]\}$ is the restriction of $\widetilde{\mathcal{F}}$ to $\Phi_j^i([\overline{\mathbb{D}}(4\varepsilon) - \mathbb{D}(3\varepsilon)] \times \mathbb{S}^1)$.

Proof. We first prove the assertion under the additional hypothesis that \mathcal{F}' is orientable, which, since it is transversely oriented, would hold if X were orientable. As \mathcal{F}' is homologically taut, then the arguments of Sullivan explained before the statement of the assertion assure that there exists a Riemannian metric $g_{\widehat{X}}$ on \widehat{X} that makes \mathcal{F}' a minimal foliation, and $g_{\widehat{X}}$ has the structure given in (17). Since \widetilde{g} is the pulled back metric of $g_{\widehat{X}}$, then the decomposition (17) can be written as well for \widetilde{g} . Our purpose is to choose the metrics g^1, g^2 in (17) (with \widetilde{g} in the left-hand-side) so that the assertion holds.

Recall that the linear map $P_\omega: T\widehat{X} \rightarrow T\widehat{\mathcal{F}}$ defined by (16) is surjective. Clearly P_ω lifts to a related surjective linear map from $T\widetilde{X}$ to $T\widetilde{\mathcal{F}}$, also denoted by P_ω . Consider a nowhere zero smooth vector field V on $\Phi_j^i([\overline{\mathbb{D}}(4\varepsilon) - \mathbb{D}(3\varepsilon)] \times \mathbb{S}^1)$ that generates the distribution $\{\ker(P_\omega)\}$. Note that V is $g_{\widehat{X}}$ -orthogonal to the leaves of $\widehat{\mathcal{F}}$ (we can consider $\Phi_j^i([\overline{\mathbb{D}}(4\varepsilon) - \mathbb{D}(3\varepsilon)] \times \mathbb{S}^1)$ to be a subset of $(\widehat{X}, g_{\widehat{X}})$ or of $(\widetilde{X}, \widetilde{g})$, and in both cases we obtain isometric compact domains with the corresponding induced metric). After multiplying V by a positive smooth function defined on $\Phi_j^i([\overline{\mathbb{D}}(4\varepsilon) - \mathbb{D}(3\varepsilon)] \times \mathbb{S}^1)$, we can assume that

$$\varphi_t[\mathbb{T}_j^i(3\varepsilon)] = \mathbb{T}_j^i(3\varepsilon + t), \quad \text{for any } t \in [0, \varepsilon], \quad (18)$$

where $\{\varphi_t\}_t$ denotes the local 1-parameter group of diffeomorphisms generated by V and $\mathbb{T}_j^i(3\varepsilon + t) = \Phi_j^i(\partial\mathbb{D}(3\varepsilon + t) \times \mathbb{S}^1)$, for each $t \in [0, \varepsilon]$, $i = 1, \dots, k$, $j = 1, 2$. Then we can identify $V = \frac{\partial}{\partial t}$. After changing the metric g^1 , we can also assume that $\frac{\partial}{\partial t}$ has g^1 -length equal to $(1 - r_2)/\varepsilon$ in $\Phi_j^i[(\mathbb{D}(4\varepsilon) - \mathbb{D}(3\varepsilon)) \times \mathbb{S}^1]$.

Two geometrical consequences of this change of metric in the $g_{\widehat{X}}$ -orthogonal direction to $T\widehat{\mathcal{F}}$ are that the distance between $\mathbb{T}_j^i(3\varepsilon)$ and $\mathbb{T}_j^i(3\varepsilon + t)$ depends linearly on $t \in [0, \varepsilon]$ and that the integral curves of $\frac{\partial}{\partial t}$ are geodesics of $g_{\widehat{X}}$. Also note that (18) implies that we can define natural “coordinates” (x, t) in $\Phi_j^i[(\mathbb{D}(4\varepsilon) - \mathbb{D}(3\varepsilon)) \times \mathbb{S}^1]$ so that $x \in \mathbb{T}_j^i(3\varepsilon)$, $t \in [0, \varepsilon]$, and we can consider the projection

$$\Pi: \mathbb{T}_j^i(3\varepsilon) \times [0, \varepsilon] \rightarrow \mathbb{T}_j^i(3\varepsilon), \quad \Pi(x, t) = x.$$

Once the above change of g^1 is done, we will change the “tangential part” g^2 of $g_{\widehat{X}}$ appropriately. Consider the induced metric $g_{3\varepsilon}$ by $g_{\widehat{X}}$ on the compact hypersurface $\mathbb{T}_j^i(3\varepsilon)$. By item (B) of Remark 4.4, we can pick a positive number λ_i so that

$$\text{Vol}(\mathbb{S}^{n-2}(r_2) \times (\mathbb{R}/\lambda_i\mathbb{Z}), g_\mu) = \text{Vol}(\mathbb{T}_j^i(3\varepsilon), g_{3\varepsilon}). \quad (19)$$

By (19) and the first theorem in Moser [10], there exists a diffeomorphism $\xi: \mathbb{T}_j^i(3\varepsilon) \rightarrow \mathbb{S}^{n-2}(r_2) \times (\mathbb{R}/\lambda_i\mathbb{Z})$ such that

$$\xi^* dV_{n-1} = \omega|_{\mathbb{T}_j^i(3\varepsilon)}, \quad (20)$$

where dV_{n-1} is the $(n-1)$ -volume form associated to the product metric g_μ defined just before Assertion 5.2.

Now consider the smooth 1-parameter family of $(n-1)$ -forms $\{\alpha_t \mid t \in [0, \varepsilon]\}$ on $\mathbb{S}^{n-2}(r_2) \times (\mathbb{R}/\lambda_i\mathbb{Z})$ defined by

$$\left[(\xi \circ \Pi)|_{\mathbb{T}_j^i(3\varepsilon) \times \{t\}} \right]^* \alpha_t = \omega|_{\mathbb{T}_j^i(3\varepsilon+t)}, \quad t \in [0, \varepsilon]. \quad (21)$$

Note that (20) and (21) imply that $\alpha_0 = dV_{n-1}$. Since ω is a closed $(n-1)$ -form, then Stokes’ theorem gives that for all $t \in [0, \varepsilon]$,

$$\begin{aligned} 0 &= \int_{\Phi_j^i[(\mathbb{D}(3\varepsilon+t) - \mathbb{D}(3\varepsilon))]} d\omega = \int_{\mathbb{T}_j^i(3\varepsilon+t)} \omega - \int_{\mathbb{T}_j^i(3\varepsilon)} \omega \\ &\stackrel{(20), (21)}{=} \int_{\mathbb{S}^{n-2}(r_2) \times (\mathbb{R}/\lambda_i\mathbb{Z})} \alpha_t - \int_{\mathbb{S}^{n-2}(r_2) \times (\mathbb{R}/\lambda_i\mathbb{Z})} dV_{n-1}. \end{aligned}$$

By Theorem 2 in [10], there exists a smooth 1-parameter family of diffeomorphisms $\phi_t: \mathbb{S}^{n-2}(r_2) \times (\mathbb{R}/\lambda_i\mathbb{Z}) \rightarrow \mathbb{S}^{n-2}(r_2) \times (\mathbb{R}/\lambda_i\mathbb{Z})$, $t \in [0, \varepsilon]$, such that ϕ_0 is the identity and

$$\phi_t^* dV_{n-1} = \alpha_t, \quad \text{for each } t \in [0, \varepsilon]. \quad (22)$$

Consider the diffeomorphism

$$\psi: \mathbb{T}_j^i(3\varepsilon) \times [0, \varepsilon] \rightarrow (\mathbb{S}^{n-2}(r_2) \times (\mathbb{R}/\lambda_i\mathbb{Z})) \times [r_2, 1], \quad \psi(x, t) = (\phi_t(\xi(x)), \frac{1-r_2}{\varepsilon}t + r_2).$$

Calling $r = \frac{1-r_2}{\varepsilon}t + r_2$ to the second variable in the target of ψ , the vector field $\frac{\partial}{\partial r}$ on $(\mathbb{S}^{n-2}(r_2) \times (\mathbb{R}/\lambda_i\mathbb{Z})) \times [r_2, 1]$ is orthogonal to the product foliation $\{[\mathbb{S}^{n-2}(r_2) \times$

$(\mathbb{R}/\lambda_i\mathbb{Z}) \times \{\delta\} \mid \delta \in [r_2, 1]\}$ with respect to the metric $g_\mu \times dr^2$. Therefore, the pullback vector field $\psi^*(\frac{\partial}{\partial r}) = (\psi^{-1})_*(\frac{\partial}{\partial r})$ is $\psi^*(g_\mu \times dr^2)$ -orthogonal to the pulled back foliation

$$\{\psi^{-1}([\mathbb{S}^{n-2}(r_2) \times (\mathbb{R}/\lambda_i\mathbb{Z})] \times \{\delta\}) \mid \delta \in [r_2, 1]\} = \{\mathbb{T}_j^i(3\varepsilon) \times \{t\} \mid t \in [0, \varepsilon]\}.$$

We claim that the $(n-1)$ -volume form of the restriction of the Riemannian metric $\psi^*(g_\mu \times dr^2)$ to the leaves of $\widehat{\mathcal{F}}$ agrees with $\omega|_{\widehat{\mathcal{F}}}$; to see this, take a $g_{3\varepsilon+t}$ -orthonormal basis $\{e_2, \dots, e_n\}$ tangent to the leaf of $\widehat{\mathcal{F}}$ passing through a point in $\mathbb{T}_j^i(3\varepsilon+t)$, $t \in [0, \varepsilon]$, positively oriented. Then,

$$\begin{aligned} 1 &= \left(\omega|_{\mathbb{T}_j^i(3\varepsilon+t)} \right) (e_2, \dots, e_n) \stackrel{(21)}{=} \alpha_t((\xi \circ \Pi)_*(e_2), \dots, (\xi \circ \Pi)_*(e_n)) \\ &\stackrel{(22)}{=} dV_{n-1}((\phi_t \circ \xi \circ \Pi)_*(e_2), \dots, (\phi_t \circ \xi \circ \Pi)_*(e_n)), \end{aligned}$$

from where our claim follows directly as dV_{n-1} is the volume form of $(\mathbb{S}^{n-2}(r_2) \times (\mathbb{R}/\lambda_i\mathbb{Z}), g_\mu)$ and the first component of ψ is $\phi_t \circ \xi \circ \Pi$ (the variable t is fixed as we are doing the computation tangentially to the leaves of $\widehat{\mathcal{F}}$).

So, after identification by ψ , Sullivan's construction can be performed with the metrics $g^1 = dr^2$ and $g^2 = g_\mu$ in $\Phi_j^i([\overline{\mathbb{D}}(4\varepsilon) - \mathbb{D}(3\varepsilon)] \times \mathbb{S}^1)$ (rigorously, we should write $g^1 = \psi^*(dr^2)$ and $g^2 = \psi^*(g_\mu)$ on $\mathbb{T}_j^i(3\varepsilon) \times [0, \varepsilon]$), and smoothly extended to the remainder of \widetilde{X} , which finishes the proof of the assertion when \mathcal{F}' is orientable.

If \mathcal{F}' is not orientable, then the above arguments can be generalized in a straightforward manner, using the covering space techniques applied in the proof of Theorem 5.3 below, to construct the desired minimal metric on \widetilde{X} . \square

To finish the proof of the existence of the ambient metric g_1 that makes \mathcal{F}' a CMC foliation, simply define g_1 to be equal to \widetilde{g} on \widetilde{X} and equal to the $(SO(n-1) \times (\mathbb{R}/\lambda_i\mathbb{Z}))$ -invariant metric defined in Section 5.1 in each of the domains $\Phi_j^i(\overline{\mathbb{D}}(3\varepsilon) \times \mathbb{S}^1)$, $i = 1, \dots, k$, $j = 1, 2$, where $\lambda_i > 0$ is defined in Assertion 5.2. We remark that the non-zero constant values H_j^i of the mean curvatures in the Reeb foliations of $\Phi_j^i(\overline{\mathbb{D}}(2\varepsilon) \times \mathbb{S}^1)$ ($H_j^i < 0$ for $j = 1$ and $H_j^i > 0$ for $j = 2$) can be chosen arbitrarily, see Remark 4.4(B).

5.5 Sullivan's theorem for non-orientable foliations.

Recall that the main theorem in Sullivan [19] asserts that an orientable foliation (of any codimension) of a compact n -manifold is geometrically taut if and only if it is homologically taut. We will next extend the implication 'homologically taut \Rightarrow geometrically taut' in Sullivan's result by dropping the orientability assumption in the case of codimension one (this extension was used in property (S4) of Section 5.4). Note that no assumption on transverse orientation is made in the foliation of the next result.

Suppose that \mathcal{F} is an oriented, smooth, k -dimensional foliation of a Riemannian n -dimensional manifold (X, g) (not necessarily $k = n - 1$), and we denote respectively by $dV_{\mathcal{F}}$, $\Pi: TX \rightarrow T\mathcal{F}$ the volume form of the induced metric by g on the leaves of \mathcal{F} and the g -orthogonal projection of TX onto $T\mathcal{F}$. In the proof of the next theorem, we will make use of the so called *Rummler's calculation*. This calculation shows that the smooth k -form ω on X given by

$$\omega_x(u_1, \dots, u_k) = dV_{\mathcal{F}}(\Pi(u_1), \dots, \Pi(u_k)), \quad x \in X, \quad u_1, \dots, u_k \in T_x X,$$

satisfies that it is \mathcal{F} -closed (i.e., the restriction of ω to every $(k+1)$ -dimensional submanifold of X tangent to \mathcal{F} is closed) if and only if the leaves of \mathcal{F} are minimal submanifolds (i.e., \mathcal{F} is geometrically taut). In particular, if $k = n - 1$, geometrically tautness of \mathcal{F} is equivalent to the fact that ω is a closed $(n - 1)$ -form on X . For details about Rummeler's calculation, see Lemma 10.5.6 in [1].

Theorem 5.3 *If a smooth codimension-one foliation of a closed n -manifold is homologically taut, then it is geometrically taut.*

Proof. Let \mathcal{F} be a codimension-one, homologically taut foliation of a closed n -manifold X . If \mathcal{F} can be oriented, then Sullivan's Theorem implies that it is geometrically taut. Suppose now that \mathcal{F} cannot be oriented. As orientability of \mathcal{F} is equivalent to orientability of the tangent subbundle $T\mathcal{F}$ of TX , then after passing to a two-sheeted cover $\Pi: \tilde{X} \rightarrow X$ of X , we can assume that the pulled back foliation $\tilde{\mathcal{F}}$ of \tilde{X} by the projection Π is orientable; this process of passing to a two-sheeted cover does not imply that \tilde{X} is orientable. Let $\sigma: \tilde{X} \rightarrow \tilde{X}$ be the order-two covering transformation. Also fix an orientation for the leaves of $\tilde{\mathcal{F}}$.

Note that since \mathcal{F} is homologically taut then $\tilde{\mathcal{F}}$ is also homologically taut. By the proof of Sullivan's Theorem, there exists a closed $(n - 1)$ -form ω on \tilde{X} that is positive on the oriented tangent spaces to the leaves of $\tilde{\mathcal{F}}$. Note that the pulled back $(n - 1)$ -form $\sigma^*\omega$ is negative on the oriented tangent spaces to the leaves of $\tilde{\mathcal{F}}$ and so, the form $\tilde{\omega} = \omega - \sigma^*(\omega)$ is positive on the tangent spaces to the leaves of $\tilde{\mathcal{F}}$. Clearly, $\tilde{\omega}$ is also closed on \tilde{X} . By the proof of Sullivan's Theorem applied to $\tilde{\omega}$, there exists a metric g' on \tilde{X} such that the $\tilde{\omega}$ restricts to be the $(n - 1)$ -volume form on the leaves of $\tilde{\mathcal{F}}$. Let $\tilde{g} = g' + \sigma^*g'$ be the related σ -invariant metric on \tilde{X} and let g be the corresponding quotient metric on X . Since $\sigma^*\tilde{\omega} = -\tilde{\omega}$, it follows that the \tilde{g} -isometry σ leaves invariant the kernel of the projection $P_{\tilde{\omega}}$, defined as in (16). In this setting, Rummeler's calculation implies that $\tilde{\mathcal{F}}$ is a minimal foliation with respect to \tilde{g} ; thus, under quotient, the foliation \mathcal{F} is minimal with respect to g , as desired. \square

Remark 5.4 With straightforward modifications, the proof of the Theorem 5.3 generalizes to the case of any smooth, k -dimensional foliation of a closed n -manifold, not just in the case that $k = n - 1$. The only difference in the proof is that after obtaining the k -form $\tilde{\omega}$, one replaces it by its purified form $\tilde{\omega}' = P_{\tilde{\omega}}^*(\tilde{\omega}|_{T\tilde{\mathcal{F}}})$ as described in page 208 of [19], which continues to be relatively \mathcal{F} -closed and to satisfy $\sigma^*\tilde{\omega}' = -\tilde{\omega}'$.

6 The proof of the Structure Theorem 1.3.

This section is devoted to the proof of Theorem 1.3. Let (X, g) be a closed, connected Riemannian n -manifold which admits a non-minimal CMC foliation \mathcal{F} . During the proof, the reader should keep in mind that since \mathcal{F} is assumed to be transversely oriented by a unit vector field $N_{\mathcal{F}}$ orthogonal to the leaves of \mathcal{F} , then every compact leaf L of \mathcal{F} is two-sided and has a closed regular neighborhood diffeomorphic to $L \times [0, 1]$. The set \mathcal{F}_0 of compact leaves of \mathcal{F} is sequentially compact in the sense that any sequence of compact leaves of \mathcal{F} has a subsequence that converges smoothly with multiplicity

one to a compact leaf of \mathcal{F} (Haefliger [5]). In particular, the union $\mathcal{C}_{\mathcal{F}}$ of the compact leaves of \mathcal{F} is a compact subset of X .

The divergence of $N_{\mathcal{F}}$ is equal to $-(n-1)H_{\mathcal{F}}$, where we are using the notation just before the statement of Theorem 1.3. Therefore, by the divergence theorem, one obtains the well-known formula

$$\int_X H_{\mathcal{F}} dV = 0,$$

which completes the proof of the first item in Theorem 1.3.

We next prove item 2 of the theorem. Since the foliation \mathcal{F} is smooth, then $H_{\mathcal{F}}: X \rightarrow \mathbb{R}$ is smooth as well. By Sard's theorem and the compactness of X , the subset of regular values of $H_{\mathcal{F}}$ is an open subset of the interval $(\min H_{\mathcal{F}}, \max H_{\mathcal{F}}) \subset \mathbb{R}$ whose complement has measure zero in $[\min H_{\mathcal{F}}, \max H_{\mathcal{F}}]$. By the implicit function theorem and the compactness of X , for each regular value H of $H_{\mathcal{F}}$, $H_{\mathcal{F}}^{-1}(H)$ consists of a finite number of compact leaves of \mathcal{F} contained in $\text{Int}(\mathcal{C}_{\mathcal{F}})$. This completes the proof of item 2.

We now proceed with the proof of item 3. Since manifolds are second countable and $\mathcal{C}_{\mathcal{F}}$ is a compact subset of X , then $X - \mathcal{C}_{\mathcal{F}}$ has a countable number of components, all of which are open. It follows that if $H_{\mathcal{F}}$ restricted to a component Δ of $X - \mathcal{C}_{\mathcal{F}}$ were not constant, then by Sard's Theorem there would be a regular value H_0 of $H_{\mathcal{F}}$ different from any of the finite number of values of $H_{\mathcal{F}}$ on the finite number of compact boundary components of Δ . By item 2 of this theorem, $\Delta \cap (H_{\mathcal{F}})^{-1}(H_0)$ contains a compact leaf of \mathcal{F} ; this is a contradiction since Δ lies in $X - \mathcal{C}_{\mathcal{F}}$. Thus, $H_{\mathcal{F}}$ is constant in every component of $X - \mathcal{C}_{\mathcal{F}}$, from which the first part of the first sentence in item 3 of the theorem follows by the continuity of $H_{\mathcal{F}}$. In particular, except for the countable subset of $H_{\mathcal{F}}(X)$ corresponding to the set of values of $H_{\mathcal{F}}$ on the components of $X - \mathcal{C}_{\mathcal{F}}$, every leaf of \mathcal{F} with mean curvature different from one of these special values is compact. Furthermore, if $H_{\Delta} \in \mathbb{R}$ is the value of $H_{\mathcal{F}}$ on a component Δ of $X - \mathcal{C}_{\mathcal{F}}$, then by the continuity of $H_{\mathcal{F}}$, H_{Δ} is the value of the mean curvature of any compact leaf in the boundary of Δ . Thus, for every $H \in H_{\mathcal{F}}(X)$ there exists at least one compact leaf of \mathcal{F} of mean curvature H .

To complete the proof of item 3, it remains to demonstrate that every leaf in the closure of $X - \mathcal{C}_{\mathcal{F}}$ is stable. To do this, first consider a component Δ of $X - \mathcal{C}_{\mathcal{F}}$. We have already proved that the leaves in the closure of Δ have the same constant mean curvature H . Since the closure of Δ in X has the structure of an H -lamination of X where every leaf is a limit leaf, then the main theorem in [9] implies that every leaf in the closure of Δ is a stable H -hypersurface. Suppose that L is a leaf in the closure of $X - \mathcal{C}_{\mathcal{F}}$ which is not a leaf in the closure of any component of $X - \mathcal{C}_{\mathcal{F}}$; in this case, every point $x \in L$ is the limit in X of a sequence of points $x_n \in X$, each of which lies in the boundary $\partial\Delta_n$ of a component Δ_n of $X - \mathcal{C}_{\mathcal{F}}$. In this case, Haefliger's compactness result for the set of closed leaves of \mathcal{F} implies that L is compact and it is the smooth limit (with multiplicity one) of a sequence of compact H_n -stable leaves $L_n \subset \partial\Delta_n$. By the continuity of $H_{\mathcal{F}}$, the H_n converge to the mean curvature of L . Since for n large any smooth unstable subdomain in L can be lifted normally to a smooth unstable subdomain in L_n , the instability of L would contradict the assumption that the L_n are H_n -stable leaves. Therefore, L must also be stable, which completes the proof of item 3 of the theorem.

To prove item 4, suppose that L is a leaf of \mathcal{F} . The local product structure of a foliation clearly implies that the set $A = \{p \in L \mid (\nabla H_{\mathcal{F}})(p) = 0\}$ is open in L . Since A is clearly closed in L and L is connected, then $A = L$ or $A = \emptyset$. In particular, if L contains a regular point of $H_{\mathcal{F}}$ then $A = L$ and so, L consists entirely of regular points of $H_{\mathcal{F}}$. By item 3, the closure of $X - \mathcal{C}_{\mathcal{F}}$ consists entirely of critical points of $H_{\mathcal{F}}$. Since every non-compact leaf of \mathcal{F} is contained in $X - \mathcal{C}_{\mathcal{F}}$, then we conclude that L must be compact. As the set of regular points of $H_{\mathcal{F}}$ is open, then the same arguments show that L lies in the interior of $\mathcal{C}_{\mathcal{F}}$ and that $\nabla H_{\mathcal{F}}$ is non-zero in a neighborhood $U(L)$ of L in X . By an application of the implicit function theorem to $H_{\mathcal{F}}$, $U(L)$ can be taken so that \mathcal{F} restricts to $U(L)$ as a product foliation of compact leaves diffeomorphic to L . This product foliation can be considered to be a smooth normal variation L_t of L through compact leaves of \mathcal{F} , whose variational field is $V = fN_{\mathcal{F}}$ for $f = g(\frac{d}{dt}|_{t=0} L_t, N_{\mathcal{F}})$ and with

$$Jf = (n-1) \frac{d}{dt} \bigg|_{t=0} H(L_t) \neq 0$$

everywhere on L , where $J = \Delta + \|\sigma\|^2 + \text{Ric}(N_{\mathcal{F}})$ is the Jacobi operator of L (here $\|\sigma\|^2$ denotes the square of the norm of the second fundamental form of L and Ric the ambient Ricci curvature). The foliation property of the variation $t \mapsto L_t$ insures that f has constant sign on L , say $f > 0$. If $Jf < 0$ on L , then Lemma 2.1 in [8] insures that L is stable (in fact, with nullity zero). Conversely, if $Jf > 0$ on L , then the index form $Q(f, f) = -\int_L f Jf$ is strictly negative, and so L is unstable. Thus, item 4a of Theorem 1.3 holds.

Suppose now that L is a leaf of \mathcal{F} that is disjoint from the regular points of $H_{\mathcal{F}}$. One possibility for L is that it is contained in the closure of $X - \mathcal{C}_{\mathcal{F}}$, in which case item 3 implies L is stable, and so, the index of L is zero. Otherwise L lies in the interior of $\mathcal{C}_{\mathcal{F}}$, which means that L is a limit leaf of the CMC lamination of X consisting of compact leaves of \mathcal{F} . As L lies in a 1-parameter family $\{L_t\}$ of compact leaves whose mean curvatures H_t at L agree up to first order with the mean curvature of L (since L is a subset of critical points of $H_{\mathcal{F}}$), the normal variational vector field F to the variation at the leaf L is a nowhere zero Jacobi field on L . It follows in this case that L is stable and has nullity one. This completes the proof of item 4b.

Finally we prove item 5. Suppose that L is a leaf of \mathcal{F} with mean curvature in $\{\min H_{\mathcal{F}}, \max H_{\mathcal{F}}\}$. Then L is stable by item 4b. Next consider a sequence of regular values r_n of $H_{\mathcal{F}}$ that are tending to, but smaller than, the value $\max H_{\mathcal{F}}$. By item 2, there exist compact leaves L_n of \mathcal{F} with $H_{\mathcal{F}}(L_n) = r_n$, and by compactness of $\mathcal{C}_{\mathcal{F}}$, after replacing by a subsequence, the L_n converge to a compact leaf L with mean curvature $\max H_{\mathcal{F}}$. Since L is disjoint from the regular values of $H_{\mathcal{F}}$ by item 4a, then the last part of item 4b implies that the nullity of L is one. A similar argument proves the existence of a compact stable leaf L' of nullity one and mean curvature $\min H_{\mathcal{F}}$, which completes the proof of Theorem 1.3.

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